Problem set 8

Problem 1. Let $X_1, X_2, \dots$ be i.i.d. Ber$(1/2)$ random variables. Fix $p > \frac{1}{2}$.

1. Show that $\mathbf{P}\{\frac{1}{n}S_n > p\} \le e^{-cn}$ for some constant $c > 0$ (that may depend on n). [*Hint:* Consider $\mathbf{E}[e^{\theta S_n}]$.]

2. Show that c may be taken to be $-p\log_2 p - (1-p)\log_2(1-p)$.

Problem 2. Let $X_1, X_2, \dots$ be i.i.d. random variables with mean μ and variance σ^2. Find a growing $[a_n, b_n]$ such that $\mathbf{P}\{a_n \le S_n \le b_n\} \to 1$ as $n \to \infty$. Try to keep the window as short as you can.

Problem 3. Find a sequence of non-negative random variables X_n such that $\mathbf{E}[X_n] \to \infty$ but $\mathbf{P}(X_n > 0) \to 0$. [*Note:* The point is that Markov's inequality cannot be reversed. Large expectation does not imply that the random variable must be large.]

Problem 4. If X is a positive random variable, show that $\mathbf{E}[\frac{1}{X}] \ge \frac{1}{\mathbf{E}[X]}$.

Problem 5. Suppose r_n labelled balls are placed uniformly at random into n distinguishable bins. Let A_n be the event that some bin is empty.

1. If $r_n = n^2$, show that $\mathbf{P}\{A_n\} \to 0$.

2. Find as small r_n as you can to get the same conclusion.

Problem 6. Suppose $X_1, X_2, \dots$ are independent but not necessarily identically distributed random variables. Assume that $\mathbf{E}[X_k] = 0$ for all k and that $\mathbf{E}[X_k^2] \le 10$ for all k. Show that $\frac{1}{n}S_n \xrightarrow{P} 0$.

Problem 7. In a population, the average height is 162cm., and the standard deviation is 9cm. Find an interval $[a, b]$ so that 90% of the population has heights between a and b.