Problem set 9

Problem 1. Let $\xi_0, \xi_1, \xi_2, \dots$ be a sequence of i.i.d. Ber(p) random variables. Define $X_0 = (\xi_0, \xi_1)$, $X_1 = (\xi_1, \xi_2)$, $\dots$. Show that (X_n) is a Markov chain with state space $\{0,1\}^2$ and find the transition matrix.

Problem 2. Let (X_n) be a Markov chain with state space S and transition matrix P. Let $Y_n = X_{2n}$ $Z_n = (X_n, X_{n+1})$, $W_n = X_{n^2}$, $V_n = \mathbf{1}_{X_n = X_{n+1}}$.

1. Show that Y and Z are Markov chains and find their state spaces and transition probabilities.
2. Show that W and V are not Markov chains in general (according to the definition given in class).

Problem 3. Let $X_0, X_1, \dots$ be a Markov chain on a state space S with transition matrix P.

1. Show that
$$\mathbf{P}\{X_0 = i, X_2 = k \,\big|\, X_1 = j\} = \mathbf{P}\{X_0 = i \,\big|\, X_1 = j\}\mathbf{P}\{X_2 = k \,\big|\, X_1 = j\}.$$
2. Generalize this to formulate and show that conditionally given X_t, the variables $(X_0, \dots, X_{t-1})$ are independent of $(X_{t+1}, \dots, X_{t+s})$.

[*Remark:* This is equivalent to Markov property. Hence, Markov property is also summarized as "independence of the past and the future, given the present".]

Problem 4. Let $(X_n)_{n \geq 0}$ be the simple symmetric random walk on $\mathbb{Z}$.

1. Explicitly write down the t-step transition probability $P^t_{i,j}$.
2. (*) Show that $\sqrt{t}P^t_{0,0}$ has a limit that is strictly positive and finite, as $t \to \infty$ *along even integers*.

Problem 5. Consider a branching process with offspring distribution having Poisson distribution with mean λ.

1. If $Z_0 = 1$, find the distribution of Z_2.

2. Find the expected value of Z_m for any m.

Problem 6. Let (X_n) be a Markov chain on the state space $\{0,1,2\dots\}$ with transition probabilities $p_{i,i+1} = \alpha$ and $p_{i,0} = 1-\alpha$, where $0 < \alpha < 1$ is fixed.

Let $T = \min\{t \geq 1 : X_t = 0\}$ (the first time the chain returns to 0). Find $P_0\{T = t\}$ for $t = 0,1,2,\dots$.

Problem 7. (*) Consider simple symmetric random walk on the n-cycle (this is the graph with vertex set $\{1,2,\dots,n\}$ with edges $1 \leftrightarrow 2,\ 2 \leftrightarrow 3,\ \dots,(n-1) \leftrightarrow n,\ n \leftrightarrow 1$).

Suppose the walk starts at 1. Let J be the last vertex that is visited by the walk. Show that J is equally likely to be any of $2,3,\dots,n$.

[*Remark:* You may assume that J is finite with probability one.]