PROBLEMS IN PROBABILITY THEORY

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Problem 1. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), under symmetric differences ($A \Delta B$).

(2) If A_n is a countable sequence of subsets of Ω, the set $\limsup_n A_n$ (respectively $\liminf_n A_n$) is defined as the set of all $\omega \in \Omega$ that belongs to infinitely many (respectively, all but finitely many) of the sets A_n.

If $A_n \in \mathcal{F}$ for all n, show that $\limsup A_n \in \mathcal{F}$ and $\liminf A_n \in \mathcal{F}$. [**Hint:** First express $\limsup A_n$ and $\liminf A_n$ in terms of A_ns and basic set operations].

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, what are $\limsup A_n$ and $\liminf A_n$?

Problem 2. Let $(\Omega, \mathcal{F})$ be a set with a σ-algebra.

(1) Suppose $\mathbf{P}$ is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and A_n increase to A (respectively, decrease to A), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P} : \mathcal{F} \to [0,1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) $\mathbf{P}$ is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and A_ns increase to A, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then, show that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

Problem 3. Suppose S is a π-system and is further closed under complements ($A \in S$ implies $A^c \in S$). Show that S is an algebra.

Problem 4. Let $\mathbf{P}$ be a p.m. on a σ-algebra $\mathcal{F}$ and suppose $S \subseteq \mathcal{F}$ be a π-system. If $A_k \in S$ for $k \leq n$, write $\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n)$ in terms of probabilities of sets in S.

Problem 5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbf{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

Problem 6. Suppose $\sigma(S) = \mathcal{F}$ and $\mathbf{P}, \mathbf{Q}$ are two probability measure on $\mathcal{F}$. If $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in S$, is it necessarily true that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$? If yes, prove it. If not, give a counterexample.

Problem 7. Let $\mathcal{F}$ be a sigma-algebra on $\mathbb{N}$ that is strictly smaller than the power set. Show that there exist $m \neq n$ such that elements of $\mathcal{F}$ do not separate m and n (i.e., any $A \in \mathcal{F}$ either contains both m, n or neither). Is the same conclusion valid if $\mathbb{N}$ is replaced by any set Ω?

Problem 8.

(1) Let $\mathcal{B}$ be the Borel sigma-algebra of $\mathbb{R}$. Show that $\mathcal{B}$ contains all closed sets, all compact sets, all intervals of the form $(a, b]$ and $[a, b)$.

(2) Show that there is a countable family $\mathcal{S}$ of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$.

(3) Let K be the $1/3$-Cantor set. Show that $\mu_*(K) = 0$.

Problem 9. Show that each of the following collection of subsets of $\mathbb{R}^d$ generate the same sigma-algebra (which we call the Borel sigma-algebra).

(1) $\{(a, b] : a < b\}$.

(2) $\{[a, b] : a \leq b \text{ and } a, b \in \mathbf{Q}\}$.

(3) The collection of all open sets.

(4) The collection of all compact sets.

Problem 10.

(1) Let X be an arbitrary set. Let S be the collection of all singletons in Ω. Describe $\sigma(S)$.

(2) Let $S = \{(a, b] \cup [-b, -a) : a < b \text{ are real numbers}\}$. Show that $\sigma(S)$ is strictly smaller than the Borel σ-algebra of $\mathbb{R}$.

(3) Suppose S is a collection of subsets of X and a, b are two elements of X such that any set in S either contains a and b both, or contains neither. Let $\mathcal{F} = \sigma(S)$. Show that any set in $\mathcal{F}$ has the same property (either contains both a and b or contains neither).

Problem 11. Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite}\}$. Define $\mu : \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite.

(1) Show that $\mathcal{A}$ is an algebra and that μ is finitely additive on $\mathcal{A}$.

(2) Under what conditions does μ extend to a probability measure on $\mathcal{F} = \sigma(\mathcal{A})$?

Problem 12. On $\mathbb{N} = \{1, 2, \ldots\}$, let A_p denote the subset of numbers divisible by p. Describe $\sigma(\{A_p : p \text{ is prime}\})$ as explicitly as possible.

Problem 13. If $\mathcal{G} \subseteq \mathcal{F}$ are sigma algebras on Ω and $\mathcal{F}$ is countably generated, then is it necessarily true that $\mathcal{G}$ is countably generated?

Problem 14. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measure spaces. If $T : X \to Y$ is a function, show that

(1) $\{T^{-1}B : B \in \mathcal{G}\}$ is a sigma algebra on X and

(2) $\{B \in \mathcal{G} : T^{-1}B \in \mathcal{F}\}$ is sigma-algebra on Y.

Problem 15. Let $A_1, A_2, \dots$ be a finite or countable partition of a non-empty set Ω (i.e., A_i are pairwise disjoint and their union is Ω). What is the σ-algebra generated by the collection of subsets $\{A_n\}$? What is the algebra generated by the same collection of subsets?

Problem 16. Let $X = [0, 1]^{\mathbb{N}}$ be the countable product of copies of $[0, 1]$. We define two sigma algebras of subsets of X.

(1) Define a metric on X by $d(x, y) = \sum_n |x_n - y_n| 2^{-n}$. Let $\mathcal{B}_X$ be the Borel sigma-algebra of (X, d). [**Note:** For those who know topology, it is better to define $\mathcal{B}_X$ as the Borel sigma algebra for the product topology on X. The point is that the metric is flexible. We can take many or other things (but not $d(x, y) = \sup_n |x_n - y_n|$!!). What matters is only the topology on X.]

(2) Let $\mathcal{C}_X$ be the sigma-algebra generated by the collection of all cylinder sets. Recall that cylinder sets are sets of the form $A = U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$ where U_i are Borel subsets of $[0, 1]$.

Show that $\mathcal{B}_X = \mathcal{C}_X$.

Problem 17. Let μ be the Lebesgue p.m. on the Cartheodary σ-algebra $\bar{\mathcal{B}}$ and let μ_* be the corresponding outer Lebesgue measure defined on all subsets of $[0, 1]$. We say that a subset $N \subseteq [0, 1]$ is a null set if $\mu_*(N) = 0$. Show that

$$\bar{\mathcal{B}} = \{B \cup N : B \in \mathcal{B} \text{ and } N \text{ is null}\}$$

where $\mathcal{B}$ is the Borel σ-algebra of $[0, 1]$.

[**Note:** The point of this exercise is to show how much larger is the Lebesgue σ-algebra than the Borel σ-algebra. The answer is, not much. Up to a null set, every Lebesgue measurable set is a Borel set. However, cardinality-wise, there is a difference. The Lebesgue σ-algebra is in bijection with $2^{\mathbb{R}}$ while the Borel σ-algebra is in bijection with $\mathbb{R}$.]

Problem 18. Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. Say that a subset $N \subseteq \Omega$ is $\mathbf{P}$-null if there exists $A \in \mathcal{F}$ with $\mathbf{P}(A) = 0$ and such that $N \subseteq A$. Define $\mathcal{G} = \{A \cup N : A \in \mathcal{F} \text{ and } N \text{ is null}\}$.

(1) Show that $\mathcal{G}$ is a σ-algebra.

(2) For $A \in \mathcal{G}$, write $A = B \cup N$ with $b \in \mathcal{F}$ and a null set N, and define $\mathbf{Q}(A) = \mathbf{P}(B)$. Show that $\mathbf{Q}$ is well-defined, that $\mathbf{Q}$ is a probability measure on $\mathcal{G}$ and $\mathbf{Q}\Big|_{\mathcal{F}} = \mathbf{P}$.

[**Note:** $\mathcal{G}$ is called the $\mathbf{P}$-completion of $\mathcal{F}$. It is a cheap way to enlarge the σ-algebra and extend the measure to the larger σ-algebra. Another description of the extended σ-algebra is $\mathcal{G} = \{A \subseteq \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subseteq A \subseteq C \text{ and } \mathbf{P}(B) = \mathbf{P}(C)\}$. Combined with the previous problem, we see that the Lebesgue σ-algebra is just the completion of the Borel σ-algebra under the Lebesgue measure. However, note that completion depends on the probability measure (for a discrete probability measure on $\mathbb{R}$, the completion will be the power set σ-algebra!). For this reason, we prefer to stick to the Borel σ-algebra and not bother to extend it.]

Problem 19. Follow these steps to obtain Sierpinski's construction of a non-measurable set. Here μ_* is the outer Lebesgue measure on $\mathbb{R}$.

(1) Regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and choose a basis H (why is it possible?).

(2) Let $A_0 = H \cup (-H) = \{x : x \in H \text{ or } -x \in H\}$. For $n \geq 1$, define $A_n := A_{n-1} - A_{n-1}$ (may also write $A_n = A_{n-1} + A_{n-1}$ since A_0 is symmetric about 0). Show that $\bigcup\limits_{n \geq 0} \bigcup\limits_{q \geq 1} \frac{1}{q} A_n = \mathbb{R}$ where $\frac{1}{q} A_n$ is the set $\{\frac{x}{q} : x \in A_n\}$.

(3) Let $N := \min\{n \geq 0 : \mu_*(A_n) > 0\}$ (you must show that N is finite!). If A_N is measurable, show that $\cup_{n \geq N+1} A_n = \mathbb{R}$.

(4) Get a contradiction to the fact that H is a basis and conclude that A_N cannot be measurable.

[**Remark:** If you start with H which has zero Lebesgue measure, then $N \geq 1$ and $A := E_{N-1}$ is a Lebesgue measurable set such that $A + A$ is not Lebesgue measurable! That was the motivation for Sierpinski. To find such a basis H, show that the Cantor set spans $\mathbb{R}$ and then choose a basis H contained inside the Cantor set.]

Problem 20. We saw that for a Borel probability measure μ on $\mathbb{R}$, the pushforward of Lebesgue measure on $[0,1]$ under the map $F_\mu^{-1} : [0,1] \to \mathbb{R}$ (as defined in lectures) is precisely μ. This is also a practical tool in simulating random variables. We assume that a random number generator gives us uniform random numbers from $[0,1]$. Apply the above idea to simulate random numbers from the following distributions (in matlab/mathematica or a program of your choice) a large number of times and compare the histogram to the actual density/mass function.

(1) Uniform distribution on $[a, b]$, (2) Exponential(λ) distribution, (3) Cauchy distribution, (4) Poisson(λ) distribution. What about the normal distribution?

Problem 21. Let $\Omega = X = \mathbb{R}$ and let $T : \Omega \to X$ be defined by $T(x) = x$. We give a pair of σ-algebras, $\mathcal{F}$ on Ω and $\mathcal{G}$ on X by taking $\mathcal{F}$ and $\mathcal{G}$ to be one of $2^{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{R}}$ or $\{\emptyset, \mathbb{R}\}$. Decide for each of the nine pairs, whether T is measurable or not.

Problem 22. (1) Define $T : \Omega \to \mathbb{R}^n$ by $T(\omega) = (\mathbf{1}_{A_1}(\omega), \ldots, \mathbf{1}_{A_n}(\omega))$ where $A_1, \ldots, A_n$ are given subsets of Ω. What is the smallest σ-algebra on Ω for which T becomes a random variable?

(2) Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and assume that $A_k \in \mathcal{F}$. Describe the push-forward measure $\mathbf{P} \circ T^{-1}$ on $\mathbb{R}^n$.

Problem 23. For $k \geq 0$, define the functions $r_k : [0,1) \to \mathbb{R}$ by writing $[0,1) = \bigsqcup_{0 \leq j < 2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k}, (j+1)2^{-k})$ and setting

$$r_k(x) = \begin{cases} -1 & \text{if } x \in I_j^{(k)} \text{ for odd } j, \\ +1 & \text{if } x \in I_j^{(k)} \text{ for even } j. \end{cases}$$

Fix $n \geq 1$ and define $T_n : [0,1) \to \{-1,1\}^n$ by $T_n(x) = (r_0(x), \ldots, r_{n-1}(x))$. Find the push-forward of the Lebesgue measure on $[0,1)$ under T_n

Problem 24. (1) If $T : \mathbb{R}^n \to \mathbb{R}^m$, show that T is Borel measurable if it is (a) continuous or (b) right continuous or (c) lower semicontinuous or (d) non-decreasing (take $m = n = 1$ for the last one).

(2) If $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with the Lebesgue sigma-algebra, show that even if T is continuous, it need not be measurable! Just do this for $n = m = 1$.

Problem 25. Show that composition of random variables is a random variable. Show that real-valued random variables on a given $(\Omega, \mathcal{F})$ are closed under linear combinations, under multiplication, under countable suprema (or infima) and under limsup (or liminf) of countable sequences.

Problem 26. Let $\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{k/n}$ and let μ be the uniform p.m. on $[0,1]$. Show directly by definition that $d(\mu_n, \mu) \to 0$ as $n \to \infty$.

Problem 27 (Change of variable for densities). (1) Let μ be a p.m. on $\mathbb{R}$ with density f by which we mean that its CDF $F_\mu(x) = \int_{-\infty}^{x} f(t)dt$ (you may assume that f is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}} f = 1$). Then, find the (density of the) push forward measure of μ under (a) $T(x) = x + a$ (b) $T(x) = bx$ (c) T is any increasing and differentiable function.

(2) If X has $N(\mu, \sigma^2)$ distribution, find the distribution of $(X - \mu)/\sigma$.

Problem 28. (1) Let $X = (X_1, \ldots, X_n)$. Show that X is an $\mathbb{R}^d$-valued r.v. if and only if $X_1, \ldots, X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1), \ldots, \sigma(X_n)$?

(2) Let $X : \Omega_1 \to \Omega_2$ be a random variable. If $X(\omega) = X(\omega')$ for some $\omega, \omega' \in \Omega_1$, show that there is no set $A \in \sigma(X)$ such that $\omega \in A$ and $\omega' \notin A$ or vice versa. [**Extra!** If $Y : \Omega_1 \to \Omega_2$ is another r.v. which is measurable w.r.t. $\sigma(X)$ on Ω_1, then show that Y is a function of X].

Problem 29 (Lévy metric). (1) Show that the Lévy metric on $\mathcal{P}(\mathbb{R}^d)$ defined in class is actually a metric.

(2) Show that under the Lévy metric, $\mathcal{P}(\mathbb{R}^d)$ is a complete and seperable metric space.

Problem 30. Let μ, ν be probability measures on $\mathbb{R}$. Let $\mathcal{C}$ be the collection of all probability measures on $\mathbb{R}^2$ whose marginals are μ and ν. Show that $\mathcal{C}$ is tight in the space of probability measures on $\mathbb{R}^2$.

Problem 31 (Lévy-Prohorov metric). If (X, d) is a metric space, let $\mathcal{P}(X)$ denote the space of Borel probability measures on X. For $\mu, \nu \in \mathcal{P}(X)$, define

$$D(\mu, \nu) = \inf\{r \geq 0 : \mu(A_r) + r \geq \nu(A) \text{ and } \nu(A_r) + r \geq \mu(A) \text{ for all closed sets } A\}.$$

Here $A_r = \{y \in X : d(x, y) \leq r \text{ for some } x \in A\}$ is the closed r-neighbourhood of A.

(1) Show that D is a metric on $\mathcal{P}(X)$.

(2) When X is $\mathbb{R}^d$, show that this agrees with the definition of Lévy metric given in class (i.e., for any μ_n, μ, we have that $\mu_n \to \mu$ in both metrics or neither).

Problem 32 (Lévy metric). Let $\mathcal{P}([-1, 1]) \subseteq \mathcal{P}(\mathbb{R})$ be the set of all Borel probability measures μ such that $\mu([-1, 1]) = 1$. For $\varepsilon > 0$, find a finite ε-net for $\mathcal{P}([-1, 1])$. [*Note:* Recall that an ε-net means a subset such that every element of $\mathcal{P}([-1, 1])$ is within ε distance of some element of the subset. Since $\mathcal{P}([-1, 1])$ is compact, we know that a finite ε-net exists for all $\varepsilon > 0$.]

Problem 33. On the probabiity space $([0, 1], \mathcal{B}, \mu)$, for $k \geq 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \geq 1$, what is the distribution of X_n?

(2) For any fixed $n \geq 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

[**Note:** $X_k(t)$ is just the k^{th} digit in the binary expansion of t. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t = 1$)].

Problem 34 (Coin tossing space). Continuing with the previous example, consider the mapping $X : [0, 1] \to \{0, 1\}^{\mathbb{N}}$ defined by $X(t) = (X_1(t), X_2(t), \ldots)$. With the Borel σ-algebra on $[0, 1]$ and the σ-algebra generated by cylinder sets on $\{0, 1\}^{\mathbb{N}}$, show that X is a random variable and find the push-foward of the Lebesue measure under X.

Problem 35 (Equivalent conditions for weak convergence). Show that the following statements are equivalent to $\mu_n \xrightarrow{d} \mu$ (you may work in $\mathcal{P}(\mathbb{R})$).

(1) $\limsup_{n\to\infty} \mu_n(F) \le \mu(F)$ if F is closed.

(2) $\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$ if G is open.

(3) $\limsup_{n\to\infty} \mu_n(A) = \mu(A)$ if $A \in \mathcal{F}$ and $\mu(\partial A) = 0$.

Problem 36. Fix $\mu \in \mathcal{P}(\mathbb{R})$. For $s \in \mathbb{R}$ and $r > 0$, let $\mu_{r,s} \in \mathcal{P}(\mathbb{R})$ be defined as $\mu_{r,s}(A) = \mu(rA + s)$ where $rA+s = \{rx+s : x \in A\}$. For which $R \subseteq (0, \infty)$ and $S \subseteq \mathbb{R}$ is it true that $\{\mu_{r,s} : r \in R,\ s \in S\}$ a tight family? [*Remark:* If not clear, just take μ to be the Lebesgue measure on $[0, 1]$.]

Problem 37. (1) Show that the family of Normal distributions $\{N(\mu, \sigma^2) : \mu \in \mathbb{R} \text{ and } \sigma^2 > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ and $B \subseteq (0, \infty)$ is the restricted family $\{N(\mu, \sigma^2) : \mu \in A \text{ and } \sigma^2 \in B\}$ tight?

Problem 38. (1) Show that the family of exponential distributions $\{\text{Exp}(\lambda) : \lambda > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ is the restricted family $\{\text{Exp}(\lambda) : \lambda > 0\}$ tight?

Problem 39. Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and that the distribution function of μ is continuous. If $\mu_n \xrightarrow{d} \mu$, show that $F_{\mu_n}(t) - F_\mu(t) \to 0$ uniformly over $t \in \mathbb{R}$. [*Restatement:* When F_μ is continuous, convergence to μ in Lévy-Prohorov metric also implies convergence in Kolmogorov-Smirnov metric.]

Problem 40. Show that the statement in the previous problem cannot be quantified. That is,

Given any $\varepsilon_n \downarrow 0$ (however fast) and $\delta_n \downarrow 0$ (however slow), show that there is some μ_n, μ with F_μ continuous, such that $d_{LP}(\mu_n, \mu) \le \varepsilon_n$ and $d_{KS}(\mu_n, \mu) \ge \delta_n$.

Problem 41. Consider the family of Normal distributions, $\{N(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$. Show that the map $(\mu, \sigma^2) \to N(\mu, \sigma^2)$ from $\mathbb{R} \times \mathbb{R}_+$ to $\mathcal{P}(\mathbb{R})$ is continuous. (Complicated way of saying that if $(\mu_n, \sigma_n^2) \to (\mu, \sigma^2)$, then $N(\mu_n, \sigma_n^2) \overset{d}{\to} N(\mu, \sigma^2)$).

Do the same for other natural families if distributions, (1) Exp(λ), (2) Uniform$[a, b]$, (3) Bin(n, p) (fix n and show continuity in p), (4) Pois(λ).

Problem 42. Suppose μ_n, μ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions $(p_n(k))_{k\in\mathbb{Z}}$ and $(p(k))_{k\in\mathbb{Z}}$. Show that $\mu_n \overset{d}{\to} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

Problem 43. Given a Borel p.m. μ on $\mathbb{R}$, show that it can be written as a convex combination $\alpha\mu_1 + (1 - \alpha)\mu_2$ with $\alpha \in [0, 1]$, where μ_1 is a purely atomic Borel p.m and μ_2 is a Borel p.m with no atoms.

Problem 44. Let F be the CDF of a Borel probability measure μ on the line.

(1) Show that F is continuous at x if and only if $\mu(\{x\}) = 0$.

(2) Show that F can have at most countably many discontinuities.

(3) Show that given any countable set $\{x_1, x_2, \ldots\}$ and any number $p_1, p_2, \ldots$ such that $\sum_i p_i \leq 1$, there is a probability measure whose CDF has a jump of magnitude p_i at x_i for each i, and no other discontinuities.

Problem 45. Let X be a random variable with distribution μ and X_n are random variables defined as follows. If μ_n is the distribution of X_n, in each case, show that $\mu_n \overset{d}{\to} \mu$ as $n \to \infty$.

(1) (Truncation). $X_n = (X \wedge n) \vee (-n)$.

(2) (Discretization). $X_n = \frac{1}{n}\lfloor nX \rfloor$.

Problem 46. Consider the space $X = [0, 1]^{\mathbb{N}} := \{\mathbf{x} = (x(1), x(2), \ldots) : 0 \leq x(i) \leq 1 \text{ for each } i \in \mathbb{N}\}$. Define the metric $d(\mathbf{x}, \mathbf{y}) = \sup_i \frac{|x(i)-y(i)|}{i}$.

(1) Show that $\mathbf{x}_n \to \mathbf{x}$ in (X, d) if and only if $x_n(i) \to x(i)$ for each i, as $n \to \infty$.

[**Note:** What matters is this pointwise convergence criterion, not the specific metric. The resulting topology is called *product topology*. The same convergence would hold if we had defined the metric as $d(\mathbf{x}, \mathbf{y}) = \sum_i 2^{-i}|x(i) - y(i)|$ or $d(\mathbf{x}, \mathbf{y}) = \sum_i i^{-2}|x(i) - y(i)|$ etc., But not the metric $\sup_i |x(i) - y(i)|$ as convergence in this metric is equivalent to uniform convergence over all $i \in \mathbb{N}$].

(2) Show that X is compact.

[**Note:** What is this problem doing here? The purpose is to reiterate a key technique we used in the proof of Helly's selection principle!]

Problem 47. Recall the Cantor set $C = \bigcap_n K_n$ where $K_0 = [0,1]$, $K_1 = [0,1/3] \cup [2/3,1]$, etc. In general, $K_n = \bigcup_{1 \le j \le 2^n} [a_{n,j}, b_{n,j}]$ where $b_{n,j} - a_{n,j} = 3^{-n}$ for each j.

(1) Let μ_n be the uniform probability measure on K_n. Describe its CDF F_n.

(2) Show that F_n converges uniformly to a CDF F.

(3) Let μ be the probability measure with CDF equal to F. Show that $\mu(C) = 1$.

Problem 48. Let $\mu \in \mathcal{P}(\mathbb{R})$.

(1) For any $n \ge 1$, define a new probability measure by $\mu_n(A) = \mu(n.A)$ where $n.A = \{nx : x \in A\}$. Does μ_n converge as $n \to \infty$?

(2) Let μ_n be defined by its CDF

$$F_n(t) = \begin{cases} 0 & \text{if } t < -n, \\ F(t) \text{ if } -n \le t < n, & \\ 1 & \text{if } t \ge n. \end{cases}$$

Does μ_n converge as $n \to \infty$?

(3) In each of the cases, describe μ_n in terms of random variables. That is, if X has distribution μ, describe a transformation $T_n(X)$ that has the distribution μ_n.

Problem 49. (Bernoulli convolutions) For any $\lambda > 1$, define $X_\lambda : [0,1] \to \mathbb{R}$ by $X(\omega) = \sum_{k=1}^{\infty} \lambda^{-k} X_k(\omega)$. Check that X_λ is measurable, and define $\mu_\lambda = \mu X_\lambda^{-1}$. Show that for any $\lambda > 2$, show that μ_λ is singular w.r.t. Lebesgue measure.

Problem 50. For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of X under the above equivalence relation).

Problem 51. (1) Give an example of a sequence of r.v.s X_n such that $\liminf \mathbf{E}[X_n] < \mathbf{E}[\liminf X_n]$.

(2) Give an example of a sequence of r.v.s X_n such that $X_n \overset{a.s.}{\to} X$, $\mathbf{E}[X_n] = 1$, but $\mathbf{E}[X] = 0$.

Problem 52. (**Alternate construction of Cantor measure**) Let $K_1 = [0,1/3] \cup [2/3,1]$, $K_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$, etc., be the decreasing sequence of compact sets whose intersection is K. Observe that K_n is a union of 2^n intervals each of length 3^{-n}. Let μ_n be the p.m. which is the "renormalized Lebesgue measure" on K_n. That is, $\mu_n(A) := 3^n 2^{-n} \mu(A \cap K_n)$ for $A \in \mathcal{B}_{\mathbb{R}}$. Then each μ_n is a Borel p.m. Show that $\mu_n \stackrel{d}{\to} \mu$, the Cantor measure (which was defined differently in class).

Problem 53. (**A quantitative characterization of absolute continuity**) Suppose $\mu \ll \nu$. Then, show that given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \varepsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

Problem 54. Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then, show that $g(x) := \int_0^x f(u)du$ is a continuous function on $[0,1]$. [**Note:** It is in fact true that g is differentiable at almost every x and that $g' = f$ $a.s.$, but that is a more sophisticated fact, called *Lebesgue's differentiation theorem*. In this course, we only need Lebesgue integration, not differentiation. The latter may be covered in your measure theory class].

Problem 55. (**Differentiating under the integral**). Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, satisfy the following assumptions.

(1) $x \to f(x,\theta)$ is Borel measurable for each θ.

(2) $\theta \to f(x,\theta)$ is continuously differentiable for each x.

(3) $f(x,\theta)$ and $\frac{\partial f}{\partial \theta}(x,\theta)$ are uniformly bounded functions of (x,θ).

Then, justify the following "differentiation under integral sign" (including the fact that the integrals here make sense).

$$\frac{d}{d\theta} \int_a^b f(x,\theta)dx = \int_a^b \frac{\partial f}{\partial \theta}(x,\theta)\, dx$$

[**Hint:** Remember that derivative is the limit of difference quotients, $h'(t) = \lim_{\varepsilon \to 0} \frac{h(t+\varepsilon) - h(t)}{\varepsilon}$.

Problem 56. (1) Let $X \geq 0$ be a r.v on $(\Omega, \mathcal{F}, \mathbf{P})$ with $0 < \mathbf{E}[X] < \infty$. Then, define $\mathbf{Q}(A) = \mathbf{E}[X\mathbf{1}_A]/\mathbf{E}[X]$ for any $A \in \mathcal{F}$. Show that $\mathbf{Q}$ is a probability measure on $\mathcal{F}$. Further, show that for any bounded random variable Y, we have $\mathbf{E}_{\mathbf{Q}}[Y] = \frac{\mathbf{E}[YX]}{\mathbf{E}[X]}$.

(2) If μ and ν are Borel probability measures on the line with continuous densities f and g (respectively) w.r.t. Lebesgue measure. Under what conditions can you assert that μ has a density w.r.t ν? In that case, what is that density?

Problem 57. For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of X under the equivalence relation $X \sim Y$ if $\mathbf{P}(X = Y) = 1$).

Problem 58. If X is an integrable random variable, show that there are bounded random variables X_n such that $\mathbf{E}[|X_n - X|] \to 0$ as $n \to \infty$.

Problem 59. Let $0 < p < q$.

(1) If $X \in L^q$, show that $X \in L^p$.

(2) If $\mathbf{E}[|X_n|^q] \to 0$ show that $\mathbf{E}[|X_n|^p] \to 0$.

Problem 60. Find integrable random variables X_n, X for each of the following situations.

(1) $X_n \to X$ a.s. but $\mathbf{E}[X_n] \not\to \mathbf{E}[X]$.

(2) $X_n \to X$ a.s. and $\mathbf{E}[X_n] \to \mathbf{E}[X]$ but there is no dominating integrable random variable Y for the sequence $\{X_n\}$.

[*Remark:* That is, the domination condition cannot be removed but can perhaps be weakened.]

Problem 61. Let X be a non-negative random variable.

(1) Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\} dt$ (in particular, if X is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$).

(2) Show that $\mathbf{E}[X^p] = \int_0^\infty p t^{p-1} \mathbf{P}\{X \geq t\} dt$ for any $p > 0$.

Problem 62. Let X be a non-negative random variable. If $\mathbf{E}[X]$ is finite, show that $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ is finite for any $a > 0$. Conversely, if $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ is finite for some $a > 0$, show that $\mathbf{E}[X]$ is finite.

Problem 63. Show that the values $\mathbf{E}[f \circ X]$ as f varies over the class of all smooth (infinitely differentiable), compactly supported functions determine the distribution of X.

Problem 64. (i) Express the mean and variance of of $aX + b$ in terms of the same quantities for X (a, b are constants).

(ii) Show that $\text{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$.

Problem 65. Compute mean, variance and moments (as many as possible!) of the Normal(0,1), exponential(1), Beta(p,q) distributions.

Problem 66. (1) Suppose $X_n \geq 0$ and $X_n \to X$ a.s. If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, show that $\mathbf{E}[|X_n - X|] \to 0$.

(2) If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{|X|>A}] \to 0$ as $A \to \infty$.

Problem 67. (1) Suppose (X,Y) has a continuous density $f(x,y)$. Find the density of X/Y. Apply to the case when (X,Y) has the *standard bivariate normal distribution* with density $f(x,y) = (2\pi)^{-1}\exp\{-\frac{x^2+y^2}{2}\}$.

(2) Find the distribution of $X+Y$ if (X,Y) has the standard bivariate normal distribution.

(3) Let $U = \min\{X,Y\}$ and $V = \max\{X,Y\}$. Find the density of (U,V).

Problem 68. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for every $f \in C_b(\mathbb{R})$. What if we only assume $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^n)$ - can we conclude that $\mu_n \xrightarrow{d} \mu$?

Problem 69. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$ having densities f_n, f with respect to Lebesgue measure. If $f_n \to f$ a.e. (w.r.t. Lebesgue measure), show that $\mu_n \xrightarrow{d} \mu$.

Problem 70 (Moment matrices). Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\alpha_k = \int x^k d\mu(x)$ (assume that all moments exist). Then, for any $n \geq 1$, show that the matrix $(\alpha_{i+j})_{0\leq i,j\leq n}$ is non-negative definite. [**Suggestion:** First solve $n = 1$].

Problem 71. Let X be a non-negative random variable with all moments (i.e., $\mathbf{E}[X^p] < \infty$ for all $p < \infty$). Show that $\log \mathbf{E}[X^p]$ is a convex function of p.

Problem 72. (1) Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^d)$. Assume that μ_n has density f_n and μ has density f w.r.t Lebesgue measure on $\mathbb{R}^n$. If $f_n(t) \to f(t)$ for all t, then show that $\mu_n \xrightarrow{d} \mu$.

(2) Show that $N(\mu_n, \sigma_n^2) \xrightarrow{d} N(\mu, \sigma)$ if and only if $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$.

Problem 73. (1) Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\alpha', 1)$ be independent random variables on a common probability space. Find the distribution of $\frac{X}{X+Y}$.

(2) If U, V are independent and have uniform([0,1]) distribution, find the distribution of $U+V$.

Problem 74. Let $\Omega = \{1, 2, \ldots, n\}$. For a probability measure $\mathbf{P}$ on Ω, we define it "entropy" $H(\mathbf{P}) := -\sum_{k=1}^{n} p_k \log p_k$ where $p_k = \mathbf{P}\{k\}$ and it is understood that $x \log x = 0$ if $x = 0$. Show that among all probability measures on Ω, the uniform probability measure (the one with $p_k = \frac{1}{n}$ for each k) is the unique maximizer of entropy.

Problem 75. (1) If $\mu_n \ll \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \ll \nu$? If $\mu_n \perp \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \perp \nu$? In either case, justify or give a counterexample.

(2) Suppose X, Y are independent (real-valued) random variables with distribution μ and ν respectively. If μ and ν are absolutely continuous w.r.t Lebesgue measure, show that the distribution of $X+Y$ is also absolutely continuous w.r.t Lebesgue measure.

Problem 76. Suppose $\{\mu_\alpha : \alpha \in I\}$ and $\{\nu_\beta : \alpha \in J\}$ are two families of Borel probability measures on $\mathbb{R}$. If both these families are tight, show that the family $\{\mu_\alpha \otimes \nu_\beta : \alpha \in I,\ \beta \in J\}$ is also tight.

Problem 77. Let X be a non-negative random variable. If $\mathbf{E}[X] \le 1$, then show that $\mathbf{E}[X^{-1}] \ge 1$.

Problem 78. Suppose X, Y are independent random variables and $X+Y$ has finite expectation. Then show that X has finite expectation. [*Hint:* Assume that Y has symmetric distribution to get a possibly simpler version of the problem]

Problem 79. On the probabiity space $([0,1], \mathcal{B}, \mu)$, for $k \ge 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup\limits_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup\limits_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \ge 1$, what is the distribution of X_n?

(2) For any fixed $n \ge 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

[**Note:** $X_k(t)$ is just the k^{th} digit in the binary expansion of t. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t = 1$)].

Problem 80. If $A \in \mathcal{B}(\mathbb{R}^2)$ has positive Lebesgue measure, show that for some $x \in \mathbb{R}$ the set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$ has positive Lebesgue measure in $\mathbb{R}$.

Problem 81 (A quantitative characterization of absolute continuity). Suppose $\mu \ll \nu$. Then, show that given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \varepsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

Problem 82. Let $Z_1, \ldots, Z_n$ be i.i.d $N(0,1)$ and write $\mathbf{Z}$ for the vector with components $Z_1, \ldots, Z_n$. Let A be an $m \times n$ matrix and let μ be a vector in $\mathbb{R}^m$. Then the m-dimensional random vector $\mathbf{X} = \mu + A\mathbf{Z}$ is said to have distribution $N_m(\mu, \Sigma)$ where $\Sigma = AA^t$ ('Normal distribution with mean vector μ and covariance matrix Σ').

(1) If $m \le n$ and A has rank m, show that $\mathbf{X}$ has density $(2\pi)^{-\frac{m}{2}} \exp\{-\frac{1}{2}\mathbf{x}^t A^{-1}\mathbf{x}\}$ w.r.t Lebesgue measure on $\mathbb{R}^m$. In particular, note that the distribution depends only on μ and AA^t. (**Note:** If $m > n$ or if rank$(A) < m$, then satisfy yourself that $\mathbf{X}$ has no density w.r.t Lebesgue measure on $\mathbb{R}^m$ - <u>you do not need to submit this</u>).

(2) Check that $\mathbf{E}[X_i] = \mu_i$ and $\text{Cov}(X_i, X_j) = \Sigma_{i,j}$.

(3) What is the distribution of (i) $(X_1, \ldots, X_k)$, for $k \le n$? (ii) $B\mathbf{X}$, where B is a $p \times m$ matrix? (iii) $X_1 + \ldots + X_m$?

Problem 83. (1) If X, Y are independent random variables, show that $\text{Cov}(X,Y) = 0$.

(2) Give a counterexample to the converse by giving an infinite sequence of random variables $X_1, X_2, \ldots$ such that $\text{Cov}(X_i, X_j) = 0$ for any $i \neq j$ but such that X_i are not independent.

(3) Suppose $(X_1, \ldots, X_m)$ has (joint) normal distribution (see the first question). If $\text{Cov}(X_i, X_j) = 0$ for all $i \le k$ and for all $j \ge k+1$, then show that $(X_1, \ldots, X_k)$ is independent of $(X_{k+1}, \ldots, X_m)$.

Problem 84. Decide whether the following are true or false and explain why.

(1) If X is independent of itself, X is constant a.s.

(2) If X is independent X^2 then X is a constant a.s.

(3) If $X, Y, X+Y$ are independent, then X and Y are constants a.s.

(4) If X and Y are independent and also $X+Y$ and $X-Y$ are independent, then X and Y must be constants a.s.

Problem 85. (1) Suppose $2 \le k < n$. Give an example of random variables $X_1, \ldots, X_n$ such that any subset of k of these random variables are independent but no subset of $k+1$ of them is independent.

(2) Suppose $(X_1, \ldots, X_n)$ has a multivariate Normal distribution. Show that if X_i are pairwise independent, then they are independent.

Problem 86. Show that it is not possible to define uncountably many independent Ber$(1/2)$ random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

Problem 87. Let $\Omega = \{1, 2, \ldots, n\}$ with the power set sigma-algebra and uniform probability measure. Let $X_p(k) = \mathbf{1}_{p \text{ divides } k}$. Are X_2 and X_3 independent? [*Note:* The answer may depend on n.]

Problem 88. Let X_i, $i \geq 1$ be random variables on a common probability space. Let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a measurable function (with product sigma algebra on $\mathbb{R}^{\mathbb{N}}$ and Borel sigma algebra on $\mathbb{R}$) and let $Y = f(X_1, X_2, \ldots)$. Show that the distribution of Y depends only on the joint distribution of $(X_1, X_2, \ldots)$ and not on the original probability space. [**Hint:** We used this to say that if X_i are independent Bernoulli random variables, then $\sum_{i \geq 1} X_i 2^{-i}$ has uniform distribution on $[0,1]$, irrespective of the underlying probability space.]

Problem 89. Let $(\Omega_1, \mathcal{F}_1, \mu)$, $(\Omega_2, \mathcal{F}_2, \nu)$ be probability spaces and let θbe a probability measure on $(\Omega = \Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. We write $z \in \Omega$ as $z = (x, y)$ (i.e., $x = \Pi_1(z)$ and $y = \Pi_2(z)$).

(1) Show that θ has marginals μ and ν if and only if,
$$\int_\Omega (f(x) + g(y)) d\theta(z) = \int_{\Omega_1} f d\mu + \int_{\Omega_2} g d\nu.$$
for every f, g bounded random variables on Ω_1 and Ω_2 respectively.

(2) Show that $\theta = \mu \otimes \nu$ if and only if
$$\int_\Omega f(x) g(y) d\theta(z) = \left(\int_{\Omega_1} f d\mu\right) \times \left(\int_{\Omega_2} g d\nu\right)$$
for every f, g bounded random variables on Ω_1 and Ω_2 respectively.

Problem 90. Suppose $(X_1, \ldots, X_n)$ has density f (w.r.t Lebesgue measure on $\mathbb{R}^n$).

(1) If $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable functions g_k, $k \leq n$. Then show that $X_1, \ldots, X_n$ are independent. (Don't assume that g_k is a density!)

(2) If $X_1, \ldots, X_n$ are independent, then $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable densities $g_1, \ldots, g_n$.

Problem 91. (1) Let S be the set of all $x \in [0,1]$ whose base b-expansion contains all the digits $0, 1, \ldots, b-1$, for every $b \in \{2, 3, 4 \ldots\}$. Show that $\lambda(S) = 1$, where λ is the Lebesgue measure on $[0,1]$.

(2) Let S be the set of all points in $\mathbb{R}^2$ that can be written as a convex combination of two rational points (a rational point is one whose co-ordinates are all rational numbers). Show that S has zero Lebesgue measure.

Problem 92. Among all $n!$ permutations of $[n]$, pick one at random with uniform probability. Show that the probability that this random permutation has no fixed points is at most $\frac{1}{2}$ for any n.

Problem 93. Suppose each of $r = \lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left(\Big| \frac{N_n}{n} - e^{-\lambda} \Big| > \delta \right) \to 0$$

Problem 94. Let X_n be i.i.d random variables such that $\mathbf{E}[|X_1|] < \infty$. Define the random power series $f(z) = \sum_{k=0}^{\infty} X_n z^n$. Show that almost surely, the radius of convergence of f is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_n z^n$ is given by $(\limsup |c_n|^{\frac{1}{n}})^{-1}$].

Problem 95. (1) Let X be a real values random variable with finite variance. Show that $f(a) := \mathbf{E}[(X-a)^2]$ is minimized at $a = \mathbf{E}[X]$.

(2) What is the quantity that minimizes $g(a) = \mathbf{E}[|X-a|]$? [**Hint:** First consider X that takes finitely many values with equal probability each].

Problem 96. If X is a positive random variable, show that $\mathbf{E}[X^p]^{\frac{1}{p}}$ is increasing in $p \in [0, \infty)$.

Problem 97. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing, continuous probability density function and let $m_p = \int_0^{\infty} x^p f(x)dx$ be its pth moment. Show that $((p+1)m_p)^{\frac{1}{p+1}}$ is increasing in $p \in [0, \infty)$.

[*Hint:* Consider a measure ν such that $\nu[x, \infty) = f(x)$ and relate m_p to ν.]

Problem 98 (Existence of Markov chains). Let S be a countable set (with the power set sigma algebra). Two ingredients are given: *A transition matrix*, that is, a function $p : S \times S \to [0,1]$ be a function such that $p(x, \cdot)$ is a probability mass function on S for each $x \in S$. (1) *An initial distribution*, that is a probability mass function μ_0 on S.

For $n \geq 0$ define the probability measure ν_n on S^{n+1} (with the product sigma algebra) by

$$\nu_n(A_0 \times A_1 \times \ldots \times A_n) = \sum_{(x_0,\ldots,x_n) \in A_0 \times \ldots \times A_n} \mu_0(x_0) \prod_{j=0}^{n-1} p(x_j, x_{j+1}).$$

Show that ν_n form a consistent family of probability distributions and conclude that a Markov chain with initial distribution μ_0 and transition matrix p exists.

Problem 99. Show that it is not possible to define uncountably many independent Ber$(1/2)$ random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

Problem 100. Let $(\Omega_i, \mathcal{F}_i, \mathbf{P}_i)$, $i \in I$, be probability spaces and let $\Omega = \times_i \Omega_i$ with $\mathcal{F} = \otimes_i \mathcal{F}_i$ and $\mathbf{P} = \otimes_i \mathbf{P}_i$. If $A \in \mathcal{F}$, show that for any $\varepsilon > 0$, there is a cylinder set B such that $\mathbf{P}(A \Delta B) < \varepsilon$.

Problem 101. Let ξ, ξ_n be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that $\limsup\limits_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1$ $a.s.$

(2) Let c_n be (non-random) complex numbers. Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} c_n \xi_n z^n$ is almost surely equal to the radius of convergence of the non-random power series $\sum_{n=0}^{\infty} c_n z^n$.

Problem 102. Let X_n be independent random variables with $X_n \sim$ Ber(p_n). For $k \geq 1$, find a sequence (p_n) so that almost surely, the sequence $X_1, X_2, \ldots$ has infinitely many segments of ones of length k but only finitely many segments of ones of length $k+1$. By a segment of length k we mean a consecutive sequence $X_i, X_{i+1}, \ldots, X_{i+k-1}$.

Problem 103. (Ergodicity of product measure). This problem guides you to a proof of a different zero-one law.

(1) Consider the product measure space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \otimes_{\mathbb{Z}} \mu)$ where $\mu \in \mathcal{P}(\mathbb{R})$. Define $\tau : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ by $(\tau\omega)_n = \omega_{n+1}$. Let $\mathcal{I} = \{A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}}) : \tau(A) = A\}$. Then, show that $\mathcal{I}$ is a sigma-algebra (called the invariant sigma algebra) and that every event in $\mathcal{I}$ has probability equal to 0 or 1.

(2) Let $X_n, n \geq 1$ be i.i.d. random variables on a common probability space. Suppose $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a measurable function such that $f(x_1, x_2, \ldots) = f(x_2, x_3, \ldots)$ for any $(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$. Then deduce from the first part that the random variable $f(X_1, X_2, \ldots)$ is a constant, $a.s.$

[**Hint:** Approximate A by cylinder sets. Use translation by τ^m to show that $\mathbf{P}(A) = \mathbf{P}(A)^2$.]

Problem 104. Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Show that there are $x_i \in \{-1, 1\}$ such that $\|x_1 v_1 + \ldots + x_n v_n\| \leq \sqrt{n}$. [*Hint:* Probabilistic method]

Problem 105. If $X \geq 0$ and $\mathbf{E}[X] = m$, then show that $\mathbf{P}\{X \leq m\} > 0$. Is there is an absolute lower bound (meaning, the bound does not depend on X) for $\mathbf{P}\{X \leq m\}$?

Problem 106. Assume $\sigma^2 := \text{Var}(X) < \infty$ and $\mathbf{E}[X] = 0$. Show that $\mathbf{P}\{X \geq t\} \leq \frac{\sigma^2}{\sigma^2+t^2}$ for $t > 0$. [*Hint:* Consider $(X - t)_{\pm}$.].

[*Note:* Compare with direct application of Chebyshev's inequality.]

Problem 107. Let X be a random variable with mean 0. Assume that $\tau = \|X\|_4$ and let $\sigma = \|X\|_2$ are finite. Let $\gamma = \tau/\sigma$. Show that

$$\mathbf{P}\{|X| \geq k\sigma\} \leq \begin{cases} \frac{1}{k^2} & \text{for any } k \geq 1, \\ \frac{\gamma^4-1}{\gamma^4+k^4-2k^2} & \text{if } k \geq \gamma^2. \end{cases}$$

[*Remark:* Strengthening of Chebyshev for high deviations, assuming 4th moment.]

Problem 108. (Chung-Erdös inequality).

(1) Let A_i be events in a probability space. Show that

$$\mathbf{P}\left\{\bigcup_{k=1}^{n} A_k\right\} \geq \frac{(\sum_{k=1}^{n} \mathbf{P}(A_k))^2}{\sum_{k,\ell=1}^{n} \mathbf{P}(A_k \cap A_\ell)}$$

(2) Place r_m balls in m bins at random and count the number of empty bins Z_m. Fix $\delta > 0$. If $r_m > (1+\delta)m \log m$, show that $\mathbf{P}(Z_m > 0) \to 0$ while if $r_m < (1-\delta)m \log m$, show that $\mathbf{P}(Z_m > 0) \to 1$.

Problem 109. Give example of an infinite sequence of pairwise independent random variables for which Kolmogorov's zero-one law fails.

Problem 110. Let X_i, $i \in I$ be random variables on a probability space. Suppose that for some $p > 0$ and $M < \infty$ we have $\mathbf{E}[|X_i|^p] \leq M$ for all $i \in I$. Show that the family $\{X_i : i \in I\}$ is tight (by which we mean that $\{\mu_{X_i} : i \in I\}$ is tight, where μ_{X_i} is the distribution of X_i).

Problem 111. Let X_i be i.i.d. random variables with zero mean and finite variance. Let $S_n = X_1 + \ldots + X_n$. Show that the collection $\{\frac{1}{\sqrt{n}} S_n : n \geq 1\}$ is tight. [**Note:** Tightness is essential for convergence in distribution. In the case at hand, convergence in distribution to $N(0,1)$ is what is called central limit theorem. We shall see it later.]

Problem 112. Suppose each of $r = \lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left(\left|\frac{N_n}{n} - e^{-\lambda}\right| > \delta\right) \to 0$$

Problem 113. Let $\xi_1, \ldots, \xi_n$ be i.i.d. tosses of a p-coin. If $\xi_{k+1} = \xi_{k+1} = \ldots = \xi_{k+m} = 1$ but $\xi_k = \xi_{k+m+1} = 0$, we say that $(k, \ldots, k+m+1)$ is a run of heads of length exactly equal to m. Let $T_{n,m}$ denote the number of runs of length exactly equal to m.

(1) For fixed m, show that $\frac{T_{n,m}}{n} \xrightarrow{P} q^2 p^m$ as $n \to \infty$.

(2) Does your proof work for $m = m_n$ increasing with n? If so how fast can it grow?

Problem 114. A random graph $\mathcal{G}_n$ with vertex set $[n] = \{1, \ldots, n\}$ is built by connecting every pair of distinct vertices with probability p_n. Show that for any $\varepsilon > 0$,

$$\mathbf{P}\{\mathcal{G}_n \text{ has an isolated vertex}\} \to \begin{cases} 1 & \text{if } p_n < (1-\varepsilon)\frac{\log n}{n} \\ 0 & \text{if } p_n > (1+\varepsilon)\frac{\log n}{n}. \end{cases}$$

[*Hint:* Consider the number of isolated vertices.]

Problem 115. Let $A_1, A_2, \ldots$ be i.i.d. uniform random subsets of $[n]$ (i.e., $\mathbf{P}(A_1 = S) = 2^{-n}$ for each $S \subseteq [n]$). Imagine sampling $A_1, A_2, \ldots$ suvvessively and let T_n be the first time when we have two subsets that are disjoint from each other. Show that $T_n \approx (2/\sqrt{3})^n$ in the sense that

$$\mathbf{P}\{T_n \geq \left(\frac{2}{\sqrt{3}}\right)^n h_n\} \to \begin{cases} 0 & \text{if } h_n \to \infty, \\ 1 & \text{if } h_n \to 0. \end{cases}$$

Problem 116. Same setting as the previous problem, but now let T_n be the first time some subset contains another. Analyse T_n as in that problem.

Problem 117. Let X_n be i.i.d random variables such that $\mathbf{E}[|X_1|] < \infty$. Define the random power series $f(z) = \sum_{k=0}^{\infty} X_n z^n$. Show that almost surely, the radius of convergence of f is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_n z^n$ is given by $(\limsup |c_n|^{\frac{1}{n}})^{-1}$].

Problem 118. Let $X_1, X_2, \ldots$ be i.i.d. fair coin tosses. Let L_n be the length of the longest run of heads in $X_1, \ldots, X_n$ (a run is a segment of consecutive tosses). Show that for any $\varepsilon > 0$,

$$\mathbf{P}\{(1-\varepsilon)\log_2 n \leq L_n \leq (1+\varepsilon)\log_2 n\} \to 1.$$

Problem 119. How does the analysis in the coupon collector problem change if one waits till each coupon is seen at least two times?

Problem 120. (1) Let X be a real values random variable with finite variance. Show that $f(a) := \mathbf{E}[(X-a)^2]$ is minimized at $a = \mathbf{E}[X]$.

(2) What is the quantity that minimizes $g(a) = \mathbf{E}[|X - a|]$? [**Hint:** First consider X that takes finitely many values with equal probability each].

Problem 121. Let X_i be i.i.d. Cauchy random variables with density $\frac{1}{\pi(1+t^2)}$. Show that $\frac{1}{n}S_n$fils the weak law of large numbers by completing the following steps.

(1) Show that $t\mathbf{P}\{|X_1| > t\} \to c$ for some constant c.

(2) Show that if $\delta > 0$ is small enough, then $\mathbf{P}\{|\frac{1}{n-1}S_{n-1}| \geq \delta\} + \mathbf{P}\{|\frac{1}{n-1}S_{n-1}| \geq \delta\}$ does not go to 0 as $n \to \infty$ [*Hint:* Consider the possibility that $|X_n| > 2\delta n$].

(3) Conclude that $\frac{1}{n}S_n \stackrel{P}{\to} 0$. [*Extra:* With a little more effort, you can try showing that there does not exist deterministic numbers a_n such that $\frac{1}{n}S_n - a_n \stackrel{P}{\to} 0$].

Problem 122. Let X_n, X be random variables on a common probability space.

(1) If $X_n \stackrel{P}{\to} X$, show that some subsequence $X_{n_k} \stackrel{a.s.}{\to} X$.

(2) If every subsequence of X_n has a further subsequence that converges almost surely to X, show that $X_n \stackrel{P}{\to} X$.

Problem 123. For $\mathbb{R}^d$-valued random vectors X_n, X, the notions of convergence almost surely, in probability and in distribution are well-defined. If $X_n = (X_{n,1}, \ldots, X_{n,d})$ and $X = (X_1, \ldots, X_d)$, which of the following is true? Justify or give counterexamples.

(1) $X_n \stackrel{a.s.}{\to} X$ if and only if $X_{n,k} \stackrel{a.s.}{\to} X_k$ for $1 \leq k \leq d$.

(2) $X_n \stackrel{P}{\to} X$ if and only if $X_{n,k} \stackrel{P}{\to} X_k$ for $1 \leq k \leq d$.

(3) $X_n \stackrel{d}{\to} X$ if and only if $X_{n,k} \stackrel{d}{\to} X_k$ for $1 \leq k \leq d$.

Problem 124. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$ (all r.v.s on the same probability space), show that $aX_n + bY_n \stackrel{P}{\to} aX + bY$ and $X_nY_n \stackrel{P}{\to} XY$. [**Hint:** You could try showing more generally that $f(X_n, Y_n) \to f(X, Y)$ for any continuous $f : \mathbb{R}^2 \to \mathbb{R}$.]

Problem 125. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) Suppose that X_n is independent of Y_n for each n (no assumptions about independence across n). If $X_n \stackrel{d}{\to} X$ and $Y_n \stackrel{d}{\to} Y$, then $(X_n, Y_n) \stackrel{d}{\to} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent. Further, $aX_n + bY_n \stackrel{d}{\to} aU + bV$.

(2) Give counterexample to show that the previous statement is false if the assumption of independence of X_n and Y_n is dropped.

Problem 126. If X_n are independent random variables and $X_n \stackrel{P}{\to} X$. Show that X is a constant random variable.

Problem 127. If X_n, Y_n are independent for each n and $X_n + Y_n \stackrel{P}{\to} 0$. Show that there are numbers y_n such that $X_n + y_n \stackrel{P}{\to} 0$.

Problem 128. Let $a_n, a \in \mathbb{R}$ and $a_n \to a$. Let $\mu_n = \frac{1}{n}(\delta_{a_1} + \ldots + \delta_{a_n})$ be the probability measure that puts mass $\frac{1}{n}$ at each a_k, $k \le n$ (with appropriate multiplicity). Show that μ_n converges in distribution and find the limit.

Problem 129. Let $\mu_n = \frac{1}{n-1}\sum_{k=1}^{n-1} \delta_{f(\frac{k}{n})}$, where $f : (0,1) \to \mathbb{R}$ is some continuous function. Show that μ_n converges in distribution and describe the limit. Find the limit explicitly when $f(x) = x^p$.

Problem 130. Suppose $\mu_n \stackrel{d}{\to} \mu$. Let $c_{n,k} \ge 0$ for $1 \le k \le n$ such that $c_{n,1} + \ldots + c_{n,n} = 1$ for each n and such that $c_{n,j} \to 0$ as $n \to \infty$ for each j. Let $\nu_n = c_{n,1}\mu_1 + \ldots + c_{n,n}\mu_n$. Show that $\nu_n \stackrel{d}{\to} \mu$.

Problem 131. For $\mathbb{R}^d$-valued random vectors X_n, X, we say that $X_n \stackrel{P}{\to} X$ if $\mathbf{P}(\|X_n - X\| > \delta) \to 0$ for any $\delta > 0$ (here you may take $\|\cdot\|$ to denote the usual norm, but any norm on $\mathbb{R}^d$ gives the same definition).

(1) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$, show that $(X_n, Y_n) \stackrel{P}{\to} (X, Y)$.

(2) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$, show that $X_n + Y_n \stackrel{P}{\to} X + Y$ and $\langle X_n, Y_n \rangle \stackrel{P}{\to} XY$. **[Hint:** Show more generally that $f(X_n, Y_n) \stackrel{P}{\to} f(X, Y)$ for any continuous function f by using the previous problem for random vectors].

Problem 132. (1) If X_n, Y_n are independent random variables on the same probability space and $X_n \stackrel{d}{\to} X$ and $Y_n \stackrel{d}{\to} Y$, then $(X_n, Y_n) \stackrel{d}{\to} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent.

(2) If $X_n \stackrel{d}{\to} X$ and $Y_n - X_n \stackrel{P}{\to} 0$, then show that $Y_n \stackrel{d}{\to} X$.

Problem 133. Let $Y_n = \frac{|X_n|}{1+|X_n|}$. Show that $X_n \stackrel{P}{\to} 0$ if and only if $Y_n \stackrel{L^1}{\to} 0$.

Problem 134. Show that the the following are equivalent conditions for tightness of a sequence $\{X_n\}$.

(1) $c_n X_n \stackrel{P}{\to} 0$ whenever $c_n \to 0$.

(2) $\mathbf{P}\{|X_n| > M_n\} \to 0$ whenever $M_n \to \infty$.

Problem 135. Show that the the following are equivalent conditions for uniform integrability of a sequence $\{X_n\}$.

(1) $c_n X_n \stackrel{L^1}{\to} 0$ whenever $c_n \to 0$.

(2) $\mathbf{E}[|X_n|\mathbf{1}_{|X_n|>M_n}] \to 0$ whenever $M_n \to \infty$.

Problem 136. For each mode of convergence (almost sure, in probability, in distribution, in L^p), decide whether the following statement is true: "If $X_n \to X$ then $\frac{1}{n}S_n \to X$", where $S_n = X_1 + \ldots + X_n$.

[*Remark:* The question is motivated by the analogous fact for convergence of numbers.]

Problem 137. Suppose X_n are i.i.d with $\mathbf{E}[|X_1|^4] < \infty$. Show that there is some constant C (depending on the distribution of X_1) such that $\mathbf{P}\left(|n^{-1}S_n - \mathbf{E}[X_1]| > \delta\right) \le Cn^{-2}$. (What is your guess if we assume $\mathbf{E}[|X_1|^6] < \infty$? You don't need to show this in the homework).

Problem 138. (1) (**Skorokhod's representation theorem**) If $X_n \stackrel{d}{\to} X$, then show that there is a probability space with random variables Y_n, Y such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ and $Y_n \stackrel{a.s.}{\to} Y$. [**Hint:** Try to construct Y_n, Y on the canonical probability space $([0,1], \mathcal{B}, \mu)$]

(2) If $X_n \stackrel{d}{\to} X$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that $f(X_n) \stackrel{d}{\to} f(X)$. [**Hint:** Use the first part]

Problem 139. Suppose X_i are i.i.d with the Cauchy distribution (density $\pi^{-1}(1+x^2)^{-1}$ on $\mathbb{R}$). Note that X_1 is not integrable. Then, show that $\frac{S_n}{n}$ does not converge in probability to any constant. [**Hint:** Try to find the probability $\mathbf{P}(X_1 > t)$, and then use it].

Problem 140. Let $U \sim$ Uniform$[0,1]$ and $X_n = \sin(nU)$. Show that X_n converges in distribution and find the limit.

Problem 141. Show that for any $p \ge 1$,

$$\lim_{n\to\infty} \int_{[0,1]^n} \frac{x_1^p + \ldots + x_n^p}{x_1 + \ldots + x_n} dx_1 \ldots dx_n = \frac{2}{p+1}.$$

[*Hint:* Do it without having to flex your muscles too much. Use probability!]

Problem 142. Let $\{X_i\}_{i\in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) If $\{X_i\}_{i\in I}$ is uniformly integrable, then show that $\sup_i \mathbf{E}|X_i| < \infty$. Give a counterexample to the converse statement.

(2) Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(x) \to \infty$ as $x \to \infty$. If $\sup_i \mathbf{E}[|X_i| h(|X_i|)] < \infty$, show that $\{X_i\}_{i\in I}$ is uniformly integrable. In particular, if $\sup_i \mathbf{E}[|X_i|^p] < \infty$ for some $p > 1$, then $\{X_i\}$ is uniformly integrable.

Problem 143. Let X_n be i.i.d with $\mathbf{P}(X_1 = +1) = \mathbf{P}(X_1 = -1) = \frac{1}{2}$. Show that for any $\gamma > \frac{1}{2}$,

$$\frac{S_n}{n^\gamma} \overset{a.s.}{\to} 0.$$

[**Remark:** Try to imitate the proof of SLLN under fourth moment assumption. If you write the proof correctly, it should go for any random variable which has moments of all orders. You do not need to show this for the homework].

Problem 144. Let X_n be independent real-valued random variables.

(1) Show by example that the event $\{\sum X_n$ converges to a number in [1,3]$\}$ can have probability strictly between 0 and 1.

(2) Show that the event $\{\sum X_n$ converges to a finite number$\}$ has probability zero or one.

Problem 145. Let X_n be i.i.d exponential(1) random variables.

(1) If b_n is a sequence of numbers that converge to 0, show that $\limsup b_n X_n$ is a constant (a.s.). Find a sequence b_n so that $\limsup\ b_n X_n = 1$ a.s.

(2) Let M_n be the maximum of $X_1, \ldots, X_n$. If $a_n \to \infty$, show that $\limsup \frac{M_n}{a_n}$ is a constant (a.s.). Find a_n so that $\limsup \frac{M_n}{a_n} = 1$ (a.s.).

[**Remark:** Can you do the same if X_n are i.i.d N(0,1)? Need not show this for the homework, but note that the main ingredient is to find a simple expression for $\mathbf{P}(X_1 > t)$ asymptotically as $t \to \infty$].

Problem 146. Let X_n be i.i.d real valued random variables with common distribution μ. For each n, define the random probabilty measure μ_n as $\mu_n := \frac{1}{n}\sum_{k=1}^n \delta_{X_k}$. Let F_n be the CDF of μ_n. Show that

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \overset{a.s.}{\to} 0 \quad a.s.$$

Problem 147. Let X_n be independent and $\mathbf{P}(X_n = n^a) = \frac{1}{2} = \mathbf{P}(X_n = -n^a)$ where $a > 0$ is fixed. For what values of a does the series $\sum X_n$ converge a.s.? For which values of a does the series converge absolutely, a.s.?

Problem 148. (**Random series**) Let X_n be i.i.d $N(0,1)$ for $n \geq 1$.

(1) Show that the random series $\sum X_n \frac{\sin(n\pi t)}{n}$ converges a.s., for any $t \in \mathbb{R}$.

(2) Show that the random series $\sum X_n \frac{t^n}{\sqrt{n!}}$ converges for all $t \in \mathbb{R}$, a.s.

[**Note:** The location of the phrase "a.s" is all important here. Let A_t and B_t denote the event that the series converges for the fixed t in the first or second parts of the question, respectively. Then, the first part is asking you to show that $\mathbf{P}(A_t) = 1$ for each $t \in \mathbb{R}$, while the second part is asking you to show that $\mathbf{P}(\cap_{t\in\mathbb{R}} B_t) = 1$. It is also true (and very important!) that $\mathbf{P}(\cap_{t\in\mathbb{R}} A_t) = 1$ but showing that is not easy.]

Problem 149. Suppose X_n are i.i.d random variables with finite mean. Which of the following assumptions guarantee that $\sum X_n$ converges a.s.?

(1) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge 1] < \infty$.

(2) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge |X_n|] < \infty$.

Problem 150. (**Large deviation for Bernoullis**). Let X_n be i.i.d Ber(1/2). Fix $p > \frac{1}{2}$.

(1) Show that $\mathbf{P}(S_n > np) \leq e^{-np\lambda} \left(\frac{e^\lambda+1}{2}\right)^n$ for any $\lambda > 0$.

(2) Optimize over λ to get $\mathbf{P}(S_n > np) \leq e^{-nI(p)}$ where $I(p) = -p\log p - (1-p)\log(1-p)$. (Observe that this is the *entropy* of the Ber(p) measure introduced in the first class test).

(3) Recall that $S_n \sim$ Binom$(n, 1/2)$, to write $\mathbf{P}(S_n = \lceil np \rceil)$ and use Stirling's approximation to show that
$$\mathbf{P}(S_n \geq np) \geq \frac{1}{\sqrt{2\pi np(1-p)}} e^{-nI(p)}.$$

(4) Deduce that $\mathbf{P}(S_n \geq np) \approx e^{-nI(p)}$ for $p > \frac{1}{2}$ and $\mathbf{P}(S_n < np) \approx e^{-nI(p)}$ for $p < \frac{1}{2}$ where the notation $a_n \approx b_n$ means $\frac{\log a_n}{\log b_n} \to 1$ as $n \to \infty$ (i.e., asymptotic equality on the logarithmic scale).

Problem 151. Carry out the same program for i.i.d exponential(1) random variables and deduce that $\mathbf{P}(S_n > nt) \approx e^{-nI(t)}$ for $t > 1$ and $\mathbf{P}(S_n < nt) \approx e^{-nI(t)}$ for $t < 1$ where $I(t) := t - 1 - \log t$.

Problem 152. Let $Y_1, \ldots, Y_n$ be independent random variables. A random variable τ taking values in $\{1, 2, \ldots, n\}$ is called a *stopping time* if the event $\{\tau \leq k\} \in \sigma(Y_1, \ldots, Y_k)$ for all k (equivalently $\{\tau = k\} \in \sigma(Y_1, \ldots, Y_k)$ for all k).

(1) Which of the following are stopping times? $\tau_1 := \min\{k \le n : S_k \in A\}$ (for some fixed $A \subseteq \mathbb{R}$). $\tau_2 := \max\{k \le n : S_k \in A\}$. $\tau_3 := \min\{k \le n : S_k = \max_{j \le n} S_j\}$. In the first two cases set $\tau = n$ if the desired event does not occur.

(2) Assuming each X_k has zero mean, show that $\mathbf{E}[S_\tau] = 0$ for any stopping time τ. Assuming that each X_k has zero mean and finite variance, show that $\mathbf{E}[S_1^2] \le \mathbf{E}[S_\tau^2] \le \mathbf{E}[S_n^2]$ for any stopping time τ.

(3) Give examples of random τ that are not stopping times and for which the results in the second part of the question fail.

Problem 153. Let X_k be independent random variables with zero mean and unit variance. Assume that $\mathbf{E}[|X_k|^{2+\delta}] \le M$ for some $\delta < 0$ and $M < \infty$. Show that S_n is asymptotically normal.

Problem 154. Let X_n be independent random variables with $X_n = \pm\sqrt{n}$ with probability $1/2$. Show that S_n satisfies the central limit theorem but not the law of large numbers.

Problem 155. Fix $\alpha > 0$.

(1) If X, Y are i.i.d. random variables such that $\frac{X+Y}{2^{\frac{1}{\alpha}}} \stackrel{d}{=} X$, then show that X must have characteristic function $\varphi_X(\lambda) = e^{-c|\lambda|^\alpha}$ for some constant c.

(2) Show that for $\alpha = 2$ we get $N(0, \sigma^2)$ and for $\alpha = 1$ we get symmetric Cauchy.

[*Note:* Only for $0 < \alpha \le 2$ is $e^{-c|\lambda|^\alpha}$ a characteristic function. Hence a distribution with the desired property exists only for this range of α].

Problem 156. Suppose X, Y are i.i.d. and $\frac{X+Y}{2^{1/\alpha}} \stackrel{d}{=} X$.

(1) If $0 < \text{Var}(X) < \infty$, show that $\alpha = 2$ and $X \sim N(0, \sigma^2)$ for some $\sigma^2 \ge 0$.

(2) If X has characteristic function $e^{-c|t|^\alpha}$ with $\alpha > 2$, deduce that $\text{Var}(X) < \infty$ and conclude that $X = 0$ (i.e., Stable-α distributions do not exist for $\alpha > 2$).

Problem 157. Let X_k be independent Ber(p_k) random variables. If $\text{Var}(S_n)$ stays bounded, show that S_n cannot be asymptotically normal.

Problem 158. Let X_n be independent random variables with zero mean and unit variance. If $\{X_n^2\}$ is uniformly integrable, show that $\frac{S_n}{\sqrt{n}} \stackrel{d}{\to} N(0, 1)$.

Problem 159. Let $U_1, U_2, \ldots$ be i.i.d. uniform$[0,1]$ random variables. Fix $0 < q < 1$ and let $M_n^{(q)}$ be the qth quantile, i.e., the $\lfloor nq \rfloor$th largest of the X_is (e.g., if $q = 1/2$, this is essentially the median). Show that $\sqrt{n}(M_n^{(q)} - q) \xrightarrow{d} N(0, q(1-q))$.

Problem 160. A simple model for grinding particles down: Start with a particle of size 1. After one cycle of grinding, it breaks into two particles of sizes X and $1 - X$, where $X \sim \mu$, a non-degenerate probability measure on $[0,1]$. Each particle of size s similarly breaks into two particles of sizes Ys and $(1-Y)s$, where $Y \sim \mu$. The random variables indicating the breaking proportion are assumed independent.

If the particle sizes are $X_{n,j}$, $j \le 2^n$, after n cycles of grinding, show that the proportion of j for which $\sqrt{n}\log X_{n,j} \le t$ converges to $\mathbf{P}\{Z \le t\}$ where $Z \sim N(0,1)$.

[*Note:* Perhaps easier, show the same for the *expected* proportion of j for which $\sqrt{n}\log X_{n,j} \le t$. This problem is a simplification of a model first proposed by Kolmogorov, where he allows each particle to subdivide into an arbitrary number of particles.]

Problem 161. Out of the $n!$ permutations of the set $[n] = \{1, 2, \ldots, n\}$, pick one at random and call it Π. Let C_n be the number of cycles in the cycle decomposition of Π.

(1) Define A_k be the event that k is the lowest element in its cycle. Show that $A_1, \ldots, A_n$ are *independent* and that $\mathbf{P}(A_k) = (n-k+1)/n$.

(2) Show that $\frac{C_n}{\log n} \xrightarrow{P} 1$.

(3) Show that $\frac{C_n - \log n}{\sqrt{\log n}} \xrightarrow{d} N(0,1)$.

Problem 162. Let X_n be independent, and let $X_n \sim (\frac{1}{2} - 2\varepsilon_n)\delta_{\pm 1} + \varepsilon_n \delta_{\pm M_n}$ where $\varepsilon_n \downarrow 0$ and $M_n \uparrow \infty$.

(1) Find a condition on M_n, ε_n that allows to apply Lindeberg-Feller theorem directly to prove that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$.

(2) If $\sum_n \varepsilon_n < \infty$, show that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$ even if M_n are chosen to violate the condition in the first part.

Problem 163. Produce an example of independent random variables X_n so that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$, but $\mathrm{Var}(S_n/\sqrt{n}) \to 2$. Can you make $\mathrm{Var}(S_n/\sqrt{n}) \to \infty$?

Problem 164 (*Weak law using characteristic functions*)**.** Let X_k be i.i.d. random variables having characteristic function φ.

(1) If $\varphi'(0) = i\mu$, show that the characteristic function of S_n/n converges to the characteristic function of δ_μ. Conclude that weak law holds for S_n/n.

(2) If $\frac{1}{n}S_n \xrightarrow{P} \mu$ for some μ, then show that φ is differentiable at 0 and $\varphi'(0) = i\mu$.

Problem 165. Find the characteristic functions of the distributions with the given densities. (1) $e^{-|x|}$ for $x \in \mathbb{R}$, (2) $\frac{1}{2}\left(1 - \frac{|x|}{2}\right)_+$.

Problem 166. Find the distributions whose characteristic functions are (1) $t \mapsto \cos(t)$, (2) $t \mapsto \frac{1}{1+it}$.

Problem 167. Show $\frac{1}{2}\text{sech}(\frac{\pi x}{2})dx$ is a probability measure whose characteristic function is $\text{sech}(t)$.

Problem 168. If $x_n \in \mathbb{R}$ and $e^{itx_n} \to 1$ for all $t \in \mathbb{R}$, then show that $x_n \to 0$.

Problem 169. If φ is a characteristic function, show that the following are also characteristic functions as a function of t. (1) $|\varphi(t)|^2$, (2) $e^{\varphi(t)-1}$, (3) $\frac{1}{t}\int_0^t \varphi(s)ds$

Problem 170. Suppose μ_n, μ are probability measures on $\mathbb{R}$ with characteristic functions φ_n, φ. If $\varphi_n(t) \to \varphi(t)$ for all $t \in \mathbb{Q}$, is it true that $\mu_n \to \mu$ weakly?

Problem 171. If ψ is a real-valued characteristic function, show that

$$1 - \psi(2t) \leq 4(1 - \psi(t)).$$

Deduce that if φ is any characteristic function, then

$$1 - |\varphi(2t)| \leq 8(1 - |\varphi(t)|).$$

Problem 172. A random variable X has characteristic function

$$\exp\left\{\sum_{j=1}^{n} \theta_j(e^{itx_j} - 1 - itx_j)\right\}$$

for some $x_i \in \mathbb{R}$ and $\theta_i > 0$. Describe/construct X in terms of familiar random variables.

Problem 173. Let $X \sim \mu$ be a random variable with characteristic function φ. Show that the following are equivalent.

(1) $X \stackrel{d}{=} Y_1 + Y_2$ for some i.i.d. random variables Y_1, Y_2.

(2) $\varphi = \psi^2$ for a characteristic function ψ.

Problem 174. Let μ be a probability measure with non-negative characteristic function $\hat{\mu} \geq 0$.

(1) If μ is supported on integers, show that $\mu\{0\} \geq \mu\{k\}$ for all $k \in \mathbb{Z}$.

(2) If $\hat{\mu}$ is integrable, show that the density of μ exists and attains its maximum at 0.

Remark (for the next three problems): The characteristic function of a $\mathbb{R}^d$-valued random vector X is the function $u \mapsto \mathbf{E}[e^{i\langle u,X\rangle}]$ from $\mathbb{R}^d \to \mathbb{C}$. Assume the following facts: If X and Y have the same characteristic functions, then $X \stackrel{d}{=} Y$. If $\mathbf{E}[e^{i\langle u,X_n\rangle}] \to \mathbf{E}[e^{i\langle u,X\rangle}]$ for all $u \in \mathbb{R}^d$, then $X_n \stackrel{d}{\to} X$.

Problem 175. Show that the measures of half-spaces (i.e., $\mathbf{P}\{\langle X, v\rangle \leq r\}$, where $v \in \mathbb{R}^d$, $r \in \mathbb{R}$) determine the distribution of X. Similarly, show that if $\langle X_n, v\rangle \stackrel{d}{\to} \langle X, v\rangle$ for each $v \in \mathbb{R}^d$, then $X_n \stackrel{d}{\to} X$.

Problem 176. If X_n are independent random vectors in $\mathbb{R}^d$ with $\mathbf{E}[X_n] = 0$ and $\mathbf{E}[X_n X_n^t] = \Sigma$, then show that $\frac{S_n}{\sqrt{n}} \stackrel{d}{\to} N_d(0, \Sigma)$, which is the defined as the distribution with the characteristic function $t \mapsto e^{-\frac{1}{2}u^t \Sigma u}$.

Problem 177. If Σ is invertible, show that $N_d(0, \Sigma)$ has density $\frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}x^t \Sigma^{-1} x}$.

Problem 178. Let X_n be i.i.d. random variables with a non-degenerate distribution. If $S_n = X_1 + \ldots + X_n$, show that $\mathbf{P}\{|S_n| \leq M\} \to 0$ for any $M < \infty$.

Problem 179. For a real valued random variable X, its *concentration function* is defined as $Q_X(t) = \sup\{\mathbf{P}\{X \in [a, a+t] : a \in \mathbb{R}\}$, for $t \geq 0$ (so $Q_X(0)$ is the largest atom size in the distribution of X). If X, Y are independent and $Z = X + Y$, show that $Q_{X+Y}(t) \leq Q_X(t)$ for all $t \geq 0$.

Problem 180. **[3 marks each]** For each of the following statements, state whether they are true or false, and justify or give counterexample accordingly.

(1) If μ, ν are Borel probability measures on $\mathbb{R}$ and $\mu \ll \nu$, then either $\nu \perp \mu$ or $\nu \ll \mu$.

(2) If $\sum\limits_n X_n$ converges a.s. and $\mathbf{P}(Y_n = X_n) = 1 - \frac{1}{n^2}$. Then $\sum\limits_n Y_n$ converges a.s.

(3) If $\{X_n\}$ is an L^2 bounded sequence of random variables, and $\mathbf{E}[X_n] = 1$ for all n, then X_n cannot converge to zero in probability.

(4) If $X_n \stackrel{d}{\to} X$, then $X_n^2 \stackrel{d}{\to} X^2$.

(5) Suppose X_n are independent with $\mathbf{E}[X_n] = 0$ and $\sum \text{Var}(X_n) = \infty$. Then, almost surely $\sum X_n$ does not converge.

(6) Suppose X_n, Y_n are random variables such that $|X_n| \le |Y_n|$ for all n. If $\sum Y_n$ converges almost surely, then $\sum X_n$ converges almost surely.

Problem 181. [2 marks+4 marks + 4 marks] Let X, Y be random variables on a common probability space. Assume that both X and Y have finite variance.

(1) Show that $\mathbf{E}[(X-a)^2]$ is minimized uniquely at $a = \mathbf{E}[X]$.

(2) Find values of a, b that minimize $f(a,b) = \mathbf{E}\left[(Y - a - bX)^2\right]$. Are they unique?

(3) Suppose $\mathbf{P}(X = k) = \frac{1}{10}$ for $k = 1, 2 \ldots, 10$. At what value(s) of a is $\mathbf{E}[|X - a|]$ minimized? Is the minimizer unique?

Problem 182. [10 marks] Let $G_1, G_2, \ldots$ be i.i.d Geometric(p) random variables (this means $\mathbf{P}(G_1 = k) = p(1-p)^{k-1}$ for $k \ge 1$). Let $X_1, X_2, \ldots$ be i.i.d random variables with $\mathbf{E}[|X_1|] < \infty$. Define $N_k := G_1 + G_2 + \ldots + G_k$. Show that as $k \to \infty$,

$$\frac{X_1 + X_2 + \ldots + X_{N_k}}{k} \xrightarrow{P} \frac{1}{p}\mathbf{E}[X_1]$$

Problem 183. [5 marks+5 marks] Let U_k, V_k be i.i.d Uniform([0,1]) random variable.

(1) Show that $\sum\limits_k U_k^{\frac{1}{k}} - V_k^{\frac{1}{k}}$ converges a.s.

(2) Let $S_n = U_1 + U_2^2 + \ldots + U_n^n$. Show that S_n satisfies a CLT. In other words, find a_n, b_n such that $\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1)$.

Problem 184. [5 marks+5 marks] Let $\mathbf{Z}^{(n)} = (Z_1^{(n)}, \ldots, Z_n^{(n)})$ be a point sampled uniformly from the sphere S^{n-1} (this means that $\mathbf{P}(\mathbf{Z}^{(n)} \in A) = \text{area}(A)/\text{area}(S^{n-1})$ for any Borel set $A \subseteq S^{n-1}$).

(1) Find the density of $Z_1^{(n)}$.

(2) Using (1) or otherwise, show that $\sqrt{n}Z_1^{(n)} \xrightarrow{d} N(0,1)$ as $n \to \infty$.

[**Hint:** One way to generate $\mathbf{Z}^{(n)}$ is to sample $X_1, \ldots, X_n$ i.i.d N(0,1) and to set $\mathbf{Z}^{(n)} = \frac{1}{\|X\|}(X_1, \ldots, X_n)$ where $\|X\| = \sqrt{X_1^2 + X_2^2 + \ldots + X_n^2}$. You may assume this fact without having to justify it].

Problem 185. [5 marks+5 marks]

(1) Let μ be a probability measure on $\mathbb{R}$ with characteristic function $\hat{\mu}(t)$. Then, show that for any $t_1, t_2, \ldots, t_n \in \mathbb{R}$, the $n \times n$ matrix A with entries $a_{i,j} = \hat{\mu}(t_i - t_j)$ is non-negative definite.

(2) Suppose $|\hat{\mu}(t_0)| = 1$ for some $t_0 \neq 0$. Then, μ is supported on a lattice, that is, $\mu(a\mathbb{Z}+b) = 1$ for some $a, b \in \mathbb{R}$. **[Hint:** Use part (1) with $n = 2$ and appropriate t_1, t_2].

Problem 186. [10 marks] Let $X_1, X_2, \ldots$ be i.i.d Bernoulli$\left(\frac{1}{2}\right)$ random variables. For each $n \geq 1$, define L_n to be the longest run of ones in $(X_1, \ldots, X_n)$, that is,

$$L_n := \max\{k : \exists j \leq n-k \text{ such that } X_{j+1} = X_{j+2} = \ldots = X_{j+k} = 1\}.$$

Prove that $\frac{L_n}{\log n} \xrightarrow{P} \frac{1}{\log 2}$.