**Homework 1: Due 27th Jan**
**Submit the first four problems only**

**1.** Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), under symmetric differences ($A \Delta B$).

(2) If $A_n$ is a countable sequence of subsets of Ω, the set $\limsup_n A_n$ (respectively $\liminf_n A_n$) is defined as the set of all $\omega \in \Omega$ that belongs to infinitely many (respectively, all but finitely many) of the sets $A_n$.

If $A_n \in \mathcal{F}$ for all $n$, show that $\limsup A_n \in \mathcal{F}$ and $\liminf A_n \in \mathcal{F}$. [**Hint:** First express $\limsup A_n$ and $\liminf A_n$ in terms of $A_n$s and basic set operations].

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, what are $\limsup A_n$ and $\liminf A_n$?

**2.** Let $(\Omega, \mathcal{F})$ be a set with a σ-algebra.

(1) Suppose **P** is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and $A_n$ increase to $A$ (respectively, decrease to $A$), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P} : \mathcal{F} \to [0,1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) **P** is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and $A_n$s increase to $A$, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then, show that **P** is a probability measure on $\mathcal{F}$.

**3.** (1) Let $\mathcal{B}$ be the Borel sigma-algebra of $\mathbb{R}$. Show that $\mathcal{B}$ contains all closed sets, all compact sets, all intervals of the form $(a,b]$ and $[a,b)$.

(2) Show that there is a countable family $\mathcal{S}$ of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$.

(3) Let $K$ be the $1/3$-Cantor set. Show that $\mu_*(K) = 0$.

**4.** (1) Let $X$ be an arbitrary set. Let $S$ be the collection of all singletons in Ω. Describe $\sigma(S)$.

(2) Let $S = \{(a,b] \cup [-b,-a) : a < b \text{ are real numbers}\}$. Show that $\sigma(S)$ is strictly smaller than the Borel σ-algebra of $\mathbb{R}$.

(3) Suppose $S$ is a collection of subsets of $X$ and $a, b$ are two elements of $X$ such that any set in $S$ either contains $a$ and $b$ both, or contains neither. Let $\mathcal{F} = \sigma(S)$. Show that any set in $\mathcal{F}$ has the same property (either contains both $a$ and $b$ or contains neither).

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**5.** Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite}\}$. Define $\mu : \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if $A$ is finite and $\mu(A) = 1$ if $A^c$ is finite.

(1) Show that $\mathcal{A}$ is an algebra and that $\mu$ is finitely additive on $\mathcal{A}$.

(2) Under what conditions does $\mu$ extend to a probability measure on $\mathcal{F} = \sigma(\mathcal{A})$?

**6.** Let $X = [0,1]^{\mathbb{N}}$ be the countable product of copies of $[0,1]$. We define two sigma algebras of subsets of $X$.

(1) Define a metric on $X$ by $d(x,y) = \sup_n |x_n - y_n|$. Let $\mathcal{B}_X$ be the Borel sigma-algebra of $(X,d)$. [**Note:** For those who know topology, it is better to define $\mathcal{B}_X$ as the Borel sigma algebra for the product topology on $X$. The point is that the metric is flexible. We can take $d(x,y) = \sum_n |x_n - y_n| 2^{-n}$ or many or other things. What matters is only the topology on $X$.]

(2) Let $\mathcal{C}_X$ be the sigma-algebra generated by the collection of all cylinder sets. Recall that cylinder sets are sets of the form $A = U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$ where $U_i$ are Borel subsets of $[0,1]$.

Show that $\mathcal{B}_X = \mathcal{C}_X$.

**7.** Let $\mu$ be the Lebesgue p.m. on the Cartheodary σ-algebra $\overline{\mathcal{B}}$ and let $\mu_*$ be the corresponding outer Lebesgue measure defined on all subsets of $[0,1]$. We say that a subset $N \subseteq [0,1]$ is a null set if $\mu_*(N) = 0$. Show that

$$\overline{\mathcal{B}} = \{B \cup N : B \in \mathcal{B} \text{ and } N \text{ is null}\}$$

where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$.

[**Note:** The point of this exercise is to show how much larger is the Lebesgue σ-algebra than the Borel σ-algebra. The answer is, not much. Up to a null set, every Lebesgue measurable set is a Borel set. However, cardinality-wise, the Lebesgue σ-algebra in bijection with $2^{\mathbb{R}}$ while the Borel σ-algebra is in bijection with $\mathbb{R}$.]