Homework 2: Due 5th Feb

Submit the first four problems only

1. (1) Let $A_1,\dots,A_n$ be subsets of Ω and define $T:\Omega\to\mathbb{R}^n$ by $T(\omega)=(\mathbf{1}_{A_1}(\omega),\dots,\mathbf{1}_{A_n}(\omega))$. What is the smallest σ-algebra on Ω for which T becomes a random variable?

(2) Suppose $(\Omega,\mathcal{F},\mathbf{P})$ is a probability space and assume that $A_k\in\mathcal{F}$. Describe the push-forward measure $\mathbf{P}\circ T^{-1}$ on $\mathbb{R}^n$.

2. (1) Let $X:\mathbb{R}^n\to\mathbb{R}^m$ be a continuous function. Show that X is a random variable. Show that X is a r.v. if it is (a) right continuous or (b) lower semicontinuous or (c) non-decreasing (take $m=n=1$ for the last one).

(2) If μ is a Borel p.m. on $\mathbb{R}$ with CDF F, then find the push-forward of μ under F.

3. For $k\ge 0$, define the functions $r_k:[0,1)\to\mathbb{R}$ by writing $[0,1)=\bigsqcup_{0\le j<2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k},(j+1)2^{-k})$ and setting

$$r_k(x)=\begin{cases}-1 & \text{if } x\in I_j^{(k)} \text{ for odd } j,\\ +1 & \text{if } x\in I_j^{(k)} \text{ for even } j.\end{cases}$$

Fix $n\ge 1$ and consider the function $T_n:[0,1)\to\{-1,1\}^n$ defined by $T_n(x)=(r_0(x),\dots,r_{n-1}(x))$. Find the pusforward of the Lebesgue measure on $[0,1)$ under T_n?

4. Let $\mu_n=\frac{1}{n}\sum_{k=1}^{n}\delta_{k/n}$ and let μ be the uniform p.m. on $[0,1]$. Show directly by definition that $d(\mu_n,\mu)\to 0$ as $n\to\infty$.

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Do not submit the following problems but highly recommended to try them

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5. Show that composition of random variables is a random variable. Show that real-valued random variables on a given $(\Omega,\mathcal{F})$ are closed under linear combinations, under multiplication, under countable suprema (or infima) and under limsup (or liminf) of countable sequences.

6. Let $\Omega=X=\mathbb{R}$ and let $T:\Omega\to X$ be defined by $T(x)=x$. We give a pair of σ-algebras, $\mathcal{F}$ on Ω and $\mathcal{G}$ on X by taking $\mathcal{F}$ and $\mathcal{G}$ to be one of $2^{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{R}}$ or $\{\emptyset,\mathbb{R}\}$. Decide for each of the nine pairs, whether T is measurable or not.

7. Follow the steps below to obtain Sierpinski's construction of a non-measurable set. Here μ_* is the outer Lebesgue measure on $\mathbb{R}$.

(1) Regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and choose a basis H (why is it possible?).

(2) Let $A_0 = H \cup (-H) = \{x : x \in H \text{ or } -x \in H\}$. For $n \geq 1$, define $A_n := A_{n-1} - A_{n-1}$ (may also write $A_n = A_{n-1} + A_{n-1}$ since A_0 is symmetric about 0). Show that $\bigcup\limits_{n\geq 0}\bigcup\limits_{q\geq 1} \frac{1}{q}A_n = \mathbb{R}$ where $\frac{1}{q}A_n$ is the set $\{\frac{x}{q} : x \in A_n\}$.

(3) Let $N := \min\{n \geq 0 : \mu_*(A_n) > 0\}$ (you must show that N is finite!). If A_N is measurable, show that $\cup_{n\geq N+1} A_n = \mathbb{R}$.

(4) Get a contradiction to the fact that H is a basis and conclude that A_N cannot be measurable.

[**Remark:** If you start with H which has zero Lebesgue measure, then $N \geq 1$ and $A := E_{N-1}$ is a Lebesgue measurable set such that $A + A$ is not Lebesgue measurable! That was the motivation for Sierpinski. To find such a basis H, show that the Cantor set spans $\mathbb{R}$ and then choose a basis H contained inside the Cantor set.]

8. We saw that for a Borel probability measure μ on $\mathbb{R}$, the pushforward of Lebesgue measure on $[0,1]$ under the map $F_\mu^{-1} : [0,1] \to \mathbb{R}$ (as defined in lectures) is precisely μ. This is also a practical tool in simulating random variables. We assume that a random number generator gives us uniform random numbers from $[0,1]$. Apply the above idea to simulate random numbers from the following distributions (in matlab/mathematica or a program of your choice) a large number of times and compare the histogram to the actual density/mass function.

(1) Uniform distribution on $[a,b]$, (2) Exponential(λ) distribution, (3) Cauchy distribution, (4) Poisson(λ) distribution. (5) What about the normal distribution?

9. Change of variable formula for densities.

(1) Let μ be a p.m. on $\mathbb{R}$ with density f by which we mean that its CDF $F_\mu(x) = \int_{-\infty}^{x} f(t)dt$ (you may assume that f is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}} f = 1$). Then, find the (density of the) push forward measure of μ under (a) $T(x) = x + a$ (b) $T(x) = bx$ (c) T is any increasing and differentiable function.

(2) If X has $N(\mu, \sigma^2)$ distribution, find the distribution of $(X-\mu)/\sigma$.

10. (1) Let $X = (X_1, \ldots, X_n)$. Show that X is an $\mathbb{R}^d$-valued r.v. if and only if $X_1, \ldots, X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1), \ldots, \sigma(X_n)$?

(2) Let $X : \Omega_1 \to \Omega_2$ be a random variable. If $X(\omega) = X(\omega')$ for some $\omega, \omega' \in \Omega_1$, show that there is no set $A \in \sigma(X)$ such that $\omega \in A$ and $\omega' \notin A$ or vice versa. [**Extra!** If $Y : \Omega_1 \to \Omega_2$ is another r.v. which is measurable w.r.t. $\sigma(X)$ on Ω_1, then show that Y is a function of X].

11. (1) Show that the Lévy metric on $\mathcal{P}(\mathbb{R}^d)$ defined in class is actually a metric.

(2) Show that under the Lévy metric, $\mathcal{P}(\mathbb{R}^d)$ is a complete and seperable metric space.