Homework 4: Due 5th Mar

Submit the first four problems only

1. (1) Suppose $X_n \geq 0$ and $X_n \to X$ a.s. If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, show that $\mathbf{E}[|X_n - X|] \to 0$.

(2) If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{|X|>A}] \to 0$ as $A \to \infty$.

2. Let X be a non-negative random variable.

(1) Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\}dt$. In particular, if X is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$.

(2) Show that $\mathbf{E}[X^p] = \int_0^\infty pt^{p-1}\mathbf{P}\{X \geq t\}dt$ for any $p > 0$.

3. Let X be a non-negative random variable with all moments (i.e., $\mathbf{E}[X^p] < \infty$ for all $p < \infty$). Show that $\log \mathbf{E}[X^p]$ is a convex function of p.

4. Compute mean and variance of the $N(0,1)$, Exp(1), and Pois(λ) distributions.

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Do not submit the following problems but highly recommended to try them

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5. (1) Give an example of a sequence of r.v.s X_n such that $\liminf \mathbf{E}[X_n] < \mathbf{E}[\liminf X_n]$.

(2) Give an example of a sequence of r.v.s X_n such that $X_n \stackrel{a.s.}{\to} X$, $\mathbf{E}[X_n] = 1$, but $\mathbf{E}[X] = 0$.

6. If $0 < p < 1$, give example to show that Minkowski's inequality may fail.

7 (Moment matrices). Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\alpha_k = \int x^k d\mu(x)$ (assume that all moments exist). Then, for any $n \geq 1$, show that the matrix $(\alpha_{i+j})_{0 \leq i,j \leq n}$ is non-negative definite. [**Suggestion:** First solve $n = 1$].

8. Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then, show that $g(x) := \int_0^x f(u)du$ is a continuous function on $[0,1]$.

[**Note:** It is in fact true that g is differentiable at almost every x and that $g' = f \quad$ a.s., but that is a more sophisticated fact, called *Lebesgue's differentiation theorem*. In this course, we only need Lebesgue integration, not differentiation. The latter may be covered in your measure theory class].

9. (**Differentiating under the integral**). Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, satisfy the following assumptions.

(1) $x \to f(x,\theta)$ is Borel measurable for each θ.

(2) $\theta \to f(x,\theta)$ is continuously differentiable for each x.

(3) $f(x,\theta)$ and $\frac{\partial f}{\partial \theta}(x,\theta)$ are uniformly bounded functions of (x,θ).

Then, justify the following "differentiation under integral sign" (including the fact that the integrals here make sense).

$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{\partial f}{\partial \theta}(x,\theta)\ dx$$

[**Hint:** Derivative is the limit of difference quotients, $h'(t) = \lim_{\epsilon\to 0} \frac{h(t+\epsilon)-h(t)}{\epsilon}$.] Contrast with the complicated conditions for the Riemann integral.