Homework 6: Due 31st Mar

Submit the first four problems only

1. (Chung-Erdös inequality). Let A_i be events in a probability space. Show that

$$\mathbf{P}\left\{\bigcup_{k=1}^{n} A_k\right\} \geq \frac{(\sum_{k=1}^{n}\mathbf{P}(A_k))^2}{\sum_{k,\ell=1}^{n}\mathbf{P}(A_k\cap A_\ell)}$$

2. (1) Suppose X_i, $i \in I$ are are random variables on a probability space and that some $p > 0$ and $M < \infty$ we have $\mathbf{E}[|X_i|^p] \leq M$ for all $i \in I$. Show that the family $\{X_i : i \in I\}$ is tight (by which we mean that $\{\mu_{X_i} : i \in I\}$ is tight, where μ_{X_i} is the distribution of X_i).

(2) Let X_i be i.i.d. random variables with zero mean and finite variance. Let $S_n = X_1 + \ldots + X_n$. Show that the collection $\{\frac{1}{\sqrt{n}}S_n : n \geq 1\}$ is tight.

3. Let ξ, ξ_n be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that $\limsup_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1$ $a.s.$

(2) Let c_n be (non-random) complex numbers. Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} c_n\xi_n z^n$ is almost surely equal to the radius of convergence of the non-random power series $\sum_{n=0}^{\infty} c_n z^n$.

4. Give example of an infinite sequence of pairwise independent random variables for which Kolmogorov's zero-one law fails.

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Do not submit the following problems but recommended to try them or at least read them!

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5. Place r_m balls in m bins at random and count the number of empty bins Z_m. Fix $\delta > 0$. If $r_m > (1+\delta)m\log m$, show that $\mathbf{P}(Z_m \geq 1) \to 0$ while if $r_m < (1-\delta)m\log m$, show that $\mathbf{P}(Z_m \geq 1) \to 1$.

6. Let $(\Omega_i, \mathcal{F}_i, \mathbf{P}_i)$, $i \in I$, be probability spaces and let $\Omega = \times_i\Omega_i$ with $\mathcal{F} = \otimes_i\mathcal{F}_i$ and $\mathbf{P} = \otimes_i\mathbf{P}_i$. If $A \in \mathcal{F}$, show that for any $\epsilon > 0$, there is a cylinder set B such that $\mathbf{P}(A\Delta B) < \epsilon$.

7. (Ergodicity of product measure). This problem guides you to a proof of a different zero-one law.

(1) Consider the product measure space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \otimes_{\mathbb{Z}}\mu)$ where $\mu \in \mathcal{P}(\mathbb{R})$. Define $\tau : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ by $(\tau\omega)_n = \omega_{n+1}$. Let $\mathcal{I} = \{A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}}) : \tau(A) = A\}$. Then, show that $\mathcal{I}$ is a sigma-algebra (called the invariant sigma algebra) and that every event in $\mathcal{I}$ has probability equal to 0 or 1.

(2) Let $X_n, n \geq 1$ be i.i.d. random variables on a common probability space. Suppose $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a measurable function such that $f(x_1, x_2, \ldots) = f(x_2, x_3, \ldots)$ for any $(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$. Then deduce from the first part that the random variable $f(X_1, X_2, \ldots)$ is a constant, $a.s.$

[**Hint:** Approximate A by cylinder sets as in the previous problem. Use translation by τ^m to prove that $\mathbf{P}(A) = \mathbf{P}(A)^2$.]

8. Consider the invariant sigma algebra and the tail sigma algebra. Show that neither is contained in the other.