HOMEWORK 7: DUE 7TH APR

SUBMIT THE FIRST FOUR PROBLEMS ONLY

1. Let X_n, X be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$, show that some subsequence $X_{n_k} \xrightarrow{a.s.} X$.

(2) If every subsequence of X_n has a further subsequence that converges almost surely to X, show that $X_n \xrightarrow{P} X$.

2. Let $\{X_i\}_{i \in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) If $\{X_i\}_{i \in I}$ is uniformly integrable, then show that $\sup_i \mathbf{E}|X_i| < \infty$. Give a counterexample to the converse statement.

(2) Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function that goes to infinity and $\sup_i \mathbf{E}[|X_i| h(|X_i|)] < \infty$. Show that $\{X_i\}_{i \in I}$ is uniformly integrable. In particular, if $\sup_i \mathbf{E}[|X_i|^p] < \infty$ for some $p > 1$, then $\{X_i\}$ is uniformly integrable.

3. Let X_n be i.i.d. $N(0,1)$ random variables. Show that $\frac{S_n}{\sqrt{n \log n}} \xrightarrow{a.s.} 0$. [*Hint:* If $\xi \sim N(0,1)$, then $\mathbf{P}\{\xi > t\} \le e^{-t^2/2}$ for $t \ge 1$. Prove this fact and use it.]

4. Let X_n be i.i.d. random variables.

(1) If $\mathbf{E}[|X_1|] < \infty$, then show that $\frac{1}{n} X_n \xrightarrow{P} 0$.

(2) Give example such that $\mathbf{E}[|X_1|] < \infty$ but $\frac{1}{n} X_n$ does not converge to 0 almost surely.

(3) Give example such that $\mathbf{E}[|X_1|] = \infty$ and $\frac{1}{n} X_n$ does not converge to 0 in probability.

[*Hint:* Relate integrability of $|X|$ to the convergence of $\sum_n \mathbf{P}\{|X| > n\}$].

———————xxxxxxxxxxxxxxxxxxxx———————

Do not submit the following problems but recommended to try them or at least read them!

———————xxxxxxxxxxxxxxxxxxxx———————

5. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (all r.v.s on the same probability space), show that $aX_n + bY_n \xrightarrow{P} aX + bY$ and $X_n Y_n \xrightarrow{P} XY$. [**Hint:** You could try showing more generally that $f(X_n, Y_n) \to f(X, Y)$ for any continuous $f : \mathbb{R}^2 \to \mathbb{R}$.]

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{d} Y$ (all on the same probability space), then show that $X_n Y_n \xrightarrow{d} XY$.

6. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) Suppose that X_n is independent of Y_n for each n (no assumptions about independence across n). If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent. Further, $aX_n + bY_n \xrightarrow{d} aU + bV$.

(2) Give counterexample to show that the previous statement is false if the assumption of independence of X_n and Y_n is dropped.

7. (1) (**Skorokhod's representation theorem**) If $X_n \xrightarrow{d} X$, then show that there is a probability space with random variables Y_n, Y such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ and $Y_n \xrightarrow{a.s.} Y$. [**Hint:** Try to construct Y_n, Y on the canonical probability space $([0,1], \mathcal{B}, \mu)$]

(2) If $X_n \xrightarrow{d} X$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that $f(X_n) \xrightarrow{d} f(X)$. [**Hint:** Use the first part]

8. Let X_i be i.i.d. Cauchy random variables with density $\frac{1}{\pi(1+t^2)}$. Show that $\frac{1}{n}S_n$ fails the weak law of large numbers by completing the following steps.

(1) Show that $t\mathbf{P}\{|X_1| > t\} \to c$ for some constant c.

(2) Show that if $\delta > 0$ is small enough, then $\mathbf{P}\{|\frac{1}{n-1}S_{n-1}| \geq \delta\} + \mathbf{P}\{|\frac{1}{n-1}S_{n-1}| \geq \delta\}$ does not go to 0 as $n \to \infty$ [*Hint:* Consider the possibility that $|X_n| > 2\delta n$].

(3) Conclude that $\frac{1}{n}S_n \overset{P}{\nrightarrow} 0$. [*Extra:* With a little more effort, you can try showing that there do not exist deterministic numbers a_n such that $\frac{1}{n}S_n - a_n \xrightarrow{P} 0$].