Problem 1:

(1) False. There are σ-algebras on $\mathbb{R}$ that are strictly smaller than $2^{\mathbb{R}}$ and that contain all singletons. Eg: (1) Borel σ-algebra
(2) $\mathcal{F} = \{A \subseteq \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable}\}$

Hence $\sigma(S) \neq 2^{\mathbb{R}}$. [Remark: In fact $\sigma(S) = \mathcal{F}$ above].

(2) False. $X = \{a, b, c\}$ $A = \{a\}$ $B = \{a, b\}$
$\mathcal{F}_1 = \{\phi, A, A^c, X\}$ $\mathcal{F}_2 = \{\phi, B, B^c, X\}$. $\mathcal{F}_1, \mathcal{F}_2$ are σ-algebras.
$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, A, B, A^c, B^c, X\}$ is not a σ-algebra since
$A, B \in \mathcal{F}_1 \cup \mathcal{F}_2$ but $A \cup B \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

(3) False. Let $\phi : [0,1] \to [0,1]$ be defined by $\phi(u) = 1-u$. Then $\lambda \circ \phi^{-1} = \lambda$.
If $\mu = N(0,1)$, we know that with $T = F_\mu^{-1}$, $\lambda \circ T^{-1} = \mu$.
Hence, $\lambda \circ (T \circ \phi)^{-1} = \lambda \circ \phi^{-1} \circ T^{-1} = \mu$ too. Uniqueness is false
[There are infinitely many $\phi : [0,1] \circlearrowleft$ that preserve λ. We can precompose T with any of them)

(4) True. If $A_0 = \phi$, there is nothing to prove. If not, let $a \in A_0$ and $t = X(a)$.
Then $X^{-1}\{t\} \in \mathcal{F}$ and $X^{-1}\{t\} \cap A_0 \neq \phi$ ($\because a \in X^{-1}\{t\} \cap A_0$)
Hence $X^{-1}\{t\} \cap A_0 = A_0$ (Note: $A_0 \in \mathcal{F}$, hence $X^{-1}\{t\} \cap A_0 \in \mathcal{F}$ too)
ie, $X(\omega) = t \ \forall t \in A_0$. $\Rightarrow X$ is a constant on A_0.

Problem 2:

(1) Let $\tilde{\mathcal{F}} = \{A \in \mathcal{B}_{\mathbb{R}} \mid X^{-1}(A) \in \mathcal{F}\}$. Then $\tilde{\mathcal{F}}$ contains all open intervals.

[$\tilde{\mathcal{F}}$ is a σ-algebra on $\mathbb{R}$]: $\phi, \mathbb{R} \in \tilde{\mathcal{F}}$ trivially. Since $X^{-1}(A^c) = (X^{-1}(A))^c$ and $X^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} X^{-1}(A_n)$, it follows that $\tilde{\mathcal{F}}$ is closed under complementation and countable unions. Hence $\tilde{\mathcal{F}}$ is a σ-algebra.

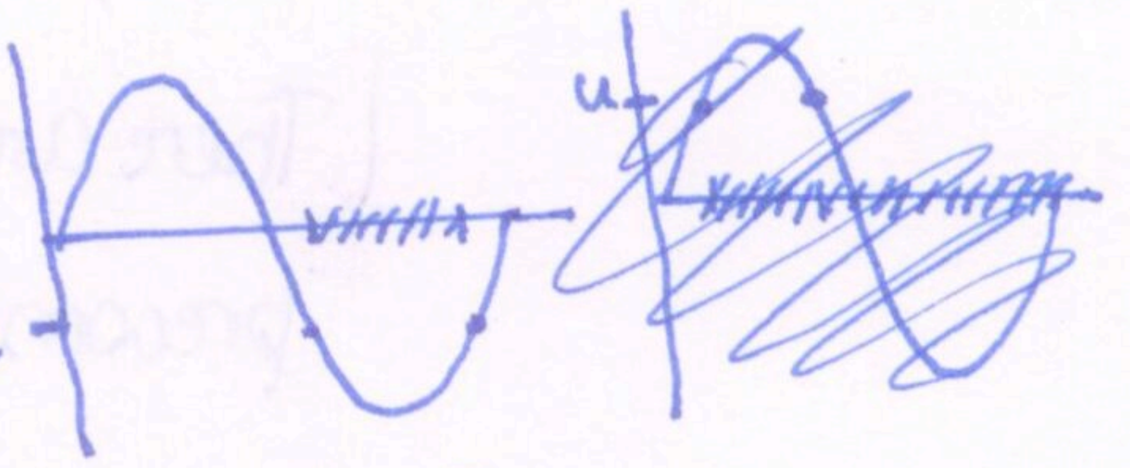
As open intervals generate $\mathcal{B}_{\mathbb{R}}$, it follows that $\mathcal{F} = \mathcal{B}_{\mathbb{R}}$. i.e. $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}_{\mathbb{R}}$.

(2) X, Y are random variables, hence $Z = (X, Y) : \Omega \to \mathbb{R}^2$ is also ~~Borel~~ measurable
$f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$ is continuous and hence Borel measurable. Therefore $f \circ Z : \Omega \to \mathbb{R}$ is a random variable, But $(f \circ Z)(\omega) = (XY)(\omega)$. Hence XY is measurable.

[<u>Alternate</u>: If $X, Y \geq 0$ then for $t < 0$ $\{XY < t\} = \phi$ while for $t > 0$ we have $\{XY < t\} = \bigcup_{q \in \mathbb{Q}} (\{X < q\} \cap \{Y < \frac{t}{q}\})$ from which it follows that XY is a r.v. Two points: (a) This needs some modification if you consider $\{XY \leq t\}$ and (b) if X and Y can take negative values]

<u>Problem 3:</u>

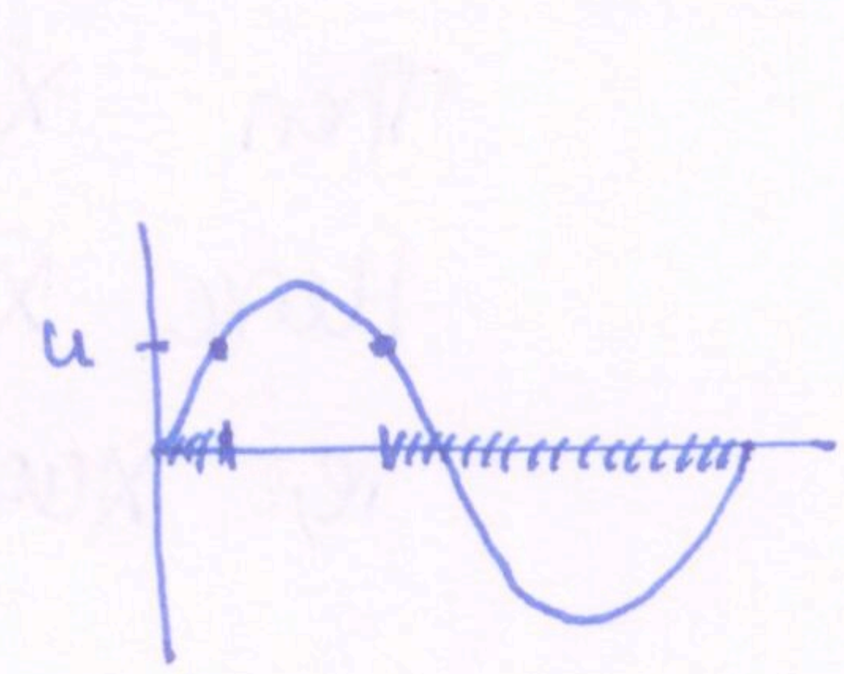
(1) Define $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ (important to specify what you mean by $\sin^{-1}$).



For $u < -1$, $\{t \mid X(t) \leq u\} = \phi$, for $u > +1$, $\{t \mid X(t) \leq u\} = [0, 1]$

For $-1 \leq u \leq 0$, $\{X \leq u\} = \left[\frac{1}{2} + \frac{\sin^{-1}(-u)}{2\pi}, 1 - \frac{\sin^{-1}(-u)}{2\pi}\right]$

For $0 \leq u \leq +1$, $\{X \leq u\} = \left[0, \frac{\sin^{-1} u}{2\pi}\right] \cup \left[\frac{1}{2} - \frac{\sin^{-1} u}{2\pi}, 1\right]$



Hence

$$F_X(u) = \lambda\{X \leq u\} = \begin{cases} 0 & \text{if } u < -1 \\ \frac{1}{2} + \frac{\sin^{-1} u}{\pi} & \text{if } -1 \leq u \leq +1 \\ 1 & \text{if } u > +1 \end{cases}$$

~~(1) False $\mathcal{B}_{\mathbb{R}}$:= Borel σ-algebra and $\mathcal{F} = \{A \mid A$ is countable~~

~~Problem 1:~~

(2) $\{Z \le u\} = \phi$ if $u<0$. For $u \geq 0$, ~~$Z(t) \le u \Leftrightarrow \lfloor \log \frac{1}{t} \rfloor \le u \Leftrightarrow t > e$~~

$$Z(t) \le u \Leftrightarrow \lfloor \log \tfrac{1}{t} \rfloor \le u$$

$$\Leftrightarrow \log \tfrac{1}{t} < \lceil u \rceil \Leftrightarrow t > e^{-\lceil u \rceil}$$

Hence $F_Z(u) = \lambda\{t \mid Z(t) \le u\} = \lambda(e^{-\lceil u \rceil}, 1] = 1 - e^{-\lceil u \rceil}$.

Thus, $F_Z(u) = \begin{cases} 0 & \text{if } u<0 \\ 1-e^{-n} & \text{if } n-1 \le u < n, \quad n=1,2,3,\dots \end{cases}$

Problem 4: Let t be a continuity point of F_μ and fix $\epsilon > 0$.

Find $\delta > 0$ such that $|F_\mu(s) - F_\mu(t)| < \epsilon$ if $|s-t| < \delta$. —(*)

Find $t_1, t_2 \in D$ such that $t-\delta < t_1 < t < t_2 < t_2+\delta$

Then, $F_{\mu_n}(t_1) \le F_{\mu_n}(t) \le F_{\mu_n}(t_2)$

$$F_\mu(t) - \epsilon \le F_\mu(t_1) \le F_\mu(t) \le F_\mu(t_2) \le F_\mu(t) + \epsilon$$

by (*) by (*)

Since $t_1, t_2 \in D$, $F_{\mu_n}(t_1) \to F_\mu(t_1)$ and $F_{\mu_n}(t_2) \to F_\mu(t_2)$.

Consequently, $\liminf F_{\mu_n}(t) \ge F_\mu(t) - \epsilon$, $\limsup F_{\mu_n}(t) \le F_\mu(t) + \epsilon$.

As ϵ is arbitrary, $F_{\mu_n}(t) \to F_\mu(t)$. Thus $\mu_n \xrightarrow{d} \mu$.

Problem 5:

(1) By definition of tightness, given $\epsilon > 0$, $\exists M$ s.t. $\forall n$ $P\{|X_n| \le M\} = \mu_{X_n}([-M, M]) \ge 1-\epsilon$ and $\forall n$ $P\{|Y_n| \le M\} = \mu_{Y_n}([-M, M]) \ge 1-\epsilon$.

Hence $P\{|Z_n| > 2M\} \le P\{|X_n| > M\} + P\{|Y_n| > M\}$ $(\because |Z_n| \le |X_n| + |Y_n|)$

$\le 2\epsilon$. (true $\forall n$).

Thus $\mu_{Z_n}([-2M, 2M]) \ge 1 - 2\epsilon$ ~~$\forall M$~~. As ϵ is arbitrary, $\{\mu_{Z_n}\}$ is tight

(2) On $([0,1], \mathcal{B}, \lambda)$, define $X_n(t) = 1$ $\forall t$ and $Y_n(t) = \frac{1}{n}$ $\forall t$. Then $W_n(t) = n$ $\forall t$.

$\mu_{X_n}([0,1]) = \mu_{Y_n}([0,1]) = 1$ so μ_{X_n}, μ_{Y_n} are tight. $\mu_{W_n}(K) \to 0$ for any compact K hence not tight.