MIDTERM 1: PROBABILITY THEORY

21ST FEB

MAXIMUM MARKS: 50, DURATION: 120 MINUTES

Note: *Give all justifications but write succinctly. Ask questions only if there is an ambiguity in the question and not to verify your answers.*

1. (**4 x 4 marks**). Declare whether the following statements are true or false and provide proofs or counterexamples accordingly.

(1) Let S be the collection of all singleton subsets of $\mathbb{R}$. Then, $\sigma(S)$ is the equal to the power set of $\mathbb{R}$.

(2) A finite union of σ-algebras is a σ-algebra.

(3) Let μ be the $N(0,1)$ distribution. Then there exists a unique measurable function $T : [0,1] \to \mathbb{R}$ such that $\lambda \circ T^{-1} = \mu$. (As usual, λ is the Lebesgue measure on $[0,1]$).

(4) Suppose $(\Omega, \mathcal{F})$ is a measurable space and $A_0 \in \mathcal{F}$ is a subset with the property that for any $A \in \mathcal{F}$, either $A \supseteq A_0$ or $A \cap A_0 = \emptyset$. Then every random variable $X : \Omega \to \mathbb{R}$ is constant on A_0.

2. (**10 marks**). Let $(\Omega, \mathcal{F})$ be a measurable space.

(1) If $X : \Omega \to \mathbb{R}$ is such that $X^{-1}(I) \in \mathcal{F}$ for every open interval I in $\mathbb{R}$, show that $X^{-1}(A) \in \mathcal{F}$ for every Borel set $A \in \mathcal{B}_{\mathbb{R}}$.

(2) If X and Y are real-valued random variables, show that XY is also a random variable.

3. (**10 marks**). Find the distribution of the following random variables defined on the probability space $([0,1], \mathcal{B}, \lambda)$. (1) $X(t) = \sin(2\pi t)$ and (2) $Z(t) = \lfloor \log \frac{1}{t} \rfloor$.

4. (**10 marks**). Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and $F_{\mu_n}(t) \to F_\mu(t)$ for all $t \in D$, where D is a dense subset of $\mathbb{R}$. Then, show that $\mu_n \xrightarrow{d} \mu$.

5. (**10 marks**). Suppose X_n, Y_n are random variables on a common probability space such that $\{\mu_{X_n}\}$ and $\{\mu_{Y_n}\}$ are tight families (here μ_X denotes the distribution of X).

(1) Let $Z_n = X_n + Y_n$ Show that $\{\mu_{Z_n}\}$ is tight.

(2) Assume that $Y_n > 0$ $a.s.$, and let $W_n = X_n / Y_n$. Show that $\{\mu_{W_n}\}$ may not be tight.