Midterm 2: Probability theory
29th March, 10AM-12:30PM
Maximum marks: 50, Duration: 150 minutes

Note: *Give all justifications but write succinctly and legibly. Ask questions only if there is an ambiguity in the question and not to verify your answers.*

1. (**10 marks**). Suppose X_n are pairwise independent (but not necessarily identically distributed) random variables on a common probability space. Assume that $\mathbf{E}[X_n] = 0$ for all n and that $|X_n| \le M$ *a.s.* for all n for some finite number M. Let $S_n = X_1 + \ldots + X_n$. Show that $\mathbf{P}\{|\frac{1}{n}S_n| \ge \delta\} \to 0$ for every $\delta > 0$.

2. (**5 x 4 marks**). Declare whether the following statements are true or false and provide proofs or counterexamples accordingly.

(1) Let $f : [0,1] \to [0,1]$ be a Borel measurable function. Define its graph as $G_f = \{(x, f(x)) : x \in [0,1]\}$. Then, $\lambda_2(G_f) = 0$, where λ_2 is the Lebesgue measure on $\mathbb{R}^2$.

(2) If $\mu \in \mathcal{P}(\mathbb{R})$ is absolutely continuous to Lebesgue measure on $\mathbb{R}$, then F_μ must be continuous.

(3) If X_i are independent random variables and $T_n = \sum_{k=1}^{n} \frac{X_k}{n-k}$. Then, $\limsup T_n$ must be constant, almost surely.

(4) If X_n are i.i.d. random variables, then $\frac{1}{n}X_n \stackrel{a.s.}{\to} 0$.

3. (**10 marks**). A box has n coupons labelled $1, 2, \ldots, n$. Coupons are drawn at random and with replacement from the box. Let T_n be the number of draws till at least 99% of coupons have been seen at least once. Show that (1) $\mathbf{E}[T_n] \sim Cn$ for some constant C and (2) $\text{Var}(T_n) = O(1)$.

4. (**10 marks**). Suppose X_n are independent random variables and $X_n \sim \text{Exp}(\lambda_n)$. Show that the series $\sum_n X_n$ converges almost surely if and only if $\sum_n \frac{1}{\lambda_n}$ is finite.

5. (**5 marks**). Let X, Y be i.i.d. random variables. If $\mathbf{P}(X = a) = 0$ for all $a \in \mathbb{R}$, then $\mathbf{P}(X = Y) = 0$.