HOMEWORK 1

Problem 1. Let $X_1, X_2, \ldots$ be i.i.d. random variables having Ber$(1/2)$ distribution. Let $S_n = X_1 + \ldots + X_n$. Find $\mathbf{E}[e^{\theta X_1}]$ for $\theta \in \mathbb{R}$ and use it to show that if $p > \frac{1}{2}$, then $\log \mathbf{P}\{S_n \geq np\} \leq -nH(p)(1+o(1))$ where $H(p) = -p\log_2 p - (1-p)\log_2(1-p)$ (it is called the *Shannon entropy* of Ber(p) distribution).

Problem 2. Let $X_1, X_2, \ldots$ be i.i.d. Exp(1) random variables. Analogous to the previous problem, show that if $t > 1$ then $\log \mathbf{P}\{S_n \geq nt\} \leq -nI(t)(1+o(1))$ where $I(t) = t\log t - t + 1$. [*Note:* For those who know the term, this is the *relative entropy* of Exp$(1/t)$ (which has mean t) w.r.t Exp(1)]

Problem 3. Let $X_1, X_2, \ldots$ be i.i.d. Exp(1). Let $M_n = \max\{X_1, \ldots, X_n\}$. Show that

(1) $\mathbf{P}\{M_n \geq (1+\delta)\log n\} \to 0$ as $n \to \infty$, for any $\delta > 0$.

(2) $\mathbf{P}\{M_n \geq (1-\delta)\log n\} \to 1$ as $n \to \infty$, for any $\delta > 0$.

[*Hint:* Fix t and consider the random variable $Z_n = \#\{k \leq n : X_k \geq t\}$ and observe that $M_n \geq t$ if and only if $Z_n > 0$]

Problem 4. Let X be a random variable with zero mean and variance σ^2. Show that $\mathbf{P}\{X \geq t\} \leq \frac{\sigma^2}{t^2+\sigma^2}$.

[*Hint:* Chebyshev's inequality for X only gives $\frac{\sigma^2}{t^2}$. Try using $X + b$ for some b]

Problem 5. Suppose r labelled balls are thrown into n labelled bins, uniformly at random and independently of each other. Let $p(r,n)$ be the probability that at least one bin is empty.

(1) Show that $p(r_n, n) \to 0$ if $\frac{r_n}{n\log n} \to \infty$.

(2) Show that $p(r_n, n) \to 1$ if $\frac{r_n}{n\log n} \to 0$.

[*Remark:* One way is to consider the number of empty bins and apply first and second moment methods.]

Problem 6. Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbf{P}\{X_1 = 0\} = 0$ (for simplicity). Show that the radius of convergence of $X_0 + X_1 z + X_2 z^2 + \ldots$ is almost surely equal to 1 if and only if $\mathbf{E}[\log_+ |X|] < \infty$ (here $\log_+ t = \max\{\log t, 0\}$ for $t > 0$).

Problem 7. Let $A_1, A_2, \ldots$ be events in a probability space. Let $Z = \sum_{k=1}^{\infty} \mathbf{1}_{A_k}$ be the number of these events that occur. Use first and second moment methods on Z to deduce Borel-Cantelli lemma in the following form:

(1) If $\sum_n \mathbf{P}(A_n) < \infty$, then almost surely, only finitely many of the A_ns occur.

(2) If A_n are *pairwise independent* and $\sum_n \mathbf{P}(A_n) = \infty$, then almost surely, infinitely many of the A_ns occur.

Problem 8. Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean μ and variance σ^2. Let $S_n = X_1 + \ldots + X_n$. Fix $\epsilon > 0$ and let A_n be the event that $|\frac{1}{n}S_n - \mu| \geq \epsilon$. Show that almost surely, only finitely many of the events A_{k^2}, $k = 1, 2, \ldots$ occur. For what other subsequences $\{n_k\}$ (in place of k^2) can you draw the same conclusion.