HOMEWORK 2

Problem 1. (1) If $X_n \xrightarrow{P} X$, show that $X_{n_k} \xrightarrow{a.s.} X$ for some subsequence.

(2) Show that $X_n \xrightarrow{P} X$ if and only if every subsequence of $\{X_n\}$ has a further subsequence that converges a.s.

Problem 2. Suppose that X_n is independent of Y_n for each n (no assumptions about independence across n). If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent. Further, $aX_n + bY_n \xrightarrow{d} aU + bV$.

Problem 3. If $\{X_i : i \in I\}$ is uniformly integrable, show that their distributions are tight. Is the converse true?

Problem 4. (1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (all r.v.s on the same probability space), show that $aX_n + bY_n \xrightarrow{P} aX + bY$ and $X_n Y_n \xrightarrow{P} XY$.

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{d} Y$ (all on the same probability space), then show that $X_n Y_n \xrightarrow{d} XY$.

Problem 5. If $\{X_i\}_{i \in I}$ and $\{Y_j\}_{j \in J}$ are both uniformly integrable families of random variables (on a common probability space), then show that $\{X_i + Y_j\}_{(i,j) \in I \times J}$ is uniformly integrable. What about the family of products, $\{X_i Y_j\}_{(i,j) \in I \times J}$?

Problem 6. For the families of Bernoulli or Normal or Exponential, determine what subsets (in terms of the parameters p or μ, σ^2 or λ) of these families are uniformly integrable.

Problem 7. Let $\{X_i : i \in I\}$ be a uniformly integrable family of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Is it true that given any $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{E}[X_i \mathbf{1}_A] < \epsilon$ whenever $\mathbf{P}(A) < \delta$?