HOMEWORK 4

PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Let $T_n = a_1X_1 + \ldots + a_nX_n$ where a_n is an increasing sequence of positive numbers.

(1) Show that T_n satisfies central limit theorem (means $(T_n - \mathbf{E}[T_n])/\sqrt{\text{Var}(T_n)} \stackrel{d}{\to} N(0,1)$) if and only if $\frac{a_n^2}{a_1^2+\ldots+a_n^2} \to 0$ as $n \to \infty$.

(2) In particular, check what happens if (a) $a_n = n^p$ for some p, (b) $a_n = e^{cn}$ for some $c > 0$, (c) $a_n = \log(n+1)$.

Problem 2. Let $X_1, X_2, \ldots$ be independent random variables with $X_n \sim \text{Ber}(p_n)$. Show that $\frac{S_n}{\sqrt{\text{Var}(S_n)}} \stackrel{d}{\to} N(0,1)$ if and only if $\text{Var}(S_n) \to \infty$.

Problem 3. Find the characteristic functions of the following distributions. (a) density $(1-|x|)$ on $[-1,1]$, (b) Poisson(λ), (c) Bin(n,p), (d) density $c\frac{\sin^2 x}{x^2}$ (where c is chosen to normalize)

Problem 4. Show that the following functions are characteristic functions and find the corresponding probability measures. (a) $\psi(t) = (1-|t|)_+$, (b) $\psi(t) = e^{-a|t|}$ where $a > 0$, (c) $\psi(t) = e^{-(at^2+bt)}$ where $a > 0$, (d) $\psi(t) = \frac{sin^k t}{t^k}$ for $k \geq 1$

Problem 5. In each of the following cases, show that ψ is a characteristic function and find the corresponding measure (in terms of measures whose characteristic functions are given).

(1) $\psi(t) = |\varphi(t)|^2$ where φ is a characteristic function.

(2) $\psi(t) = p_1\varphi_1(t) + \ldots + p_n\varphi_n(t)$ where φ_i are characteristic functions and $p_i \geq 0$ add up to 1.

(3) $\psi(t) = Q(\varphi(t))$ where φ is a characteristic function and $Q(x) = q_kx^k + \ldots + q_1x + q_0$ where $q_i \geq 0$ and $q_0 + \ldots + q_k = 1$.

Problem 6. Let $X_{n,k}$, $1 \leq k \leq n$, be a triangular array of Bernoulli random variables such that (a) $X_{n,1}, \ldots, X_{n,n}$ are independent for each n, (b) $X_{n,k} \sim \text{Ber}(p_{n,k})$, (c) $p_{n,1} + \ldots + p_{n,n} \to \lambda \in (0,\infty)$ as $n \to \infty$, and $\max\{p_{n,1}, \ldots, p_{n,n}\} \to 0$ as $n \to \infty$. Show that $S_n \stackrel{d}{\to} \text{Pois}(\lambda)$.

Problem 7. Suppose (x_n) is a sequence of real numbers such that $e^{itx_n} \to 1$ as $n \to \infty$, for any $t \in \mathbb{R}$. Show that $x_n \to 0$. [*Note:* This is just for entertainment. You can get it from Lévy's continuity theorem. Can you get a proof without using any such machinery?]

Problem 8. (*) Let C_n be the number of cycles of a uniformly chosen random permutation of $\{1, 2, \ldots, n\}$. Show that C_n satisfies a CLT (give the precise statement).