Solutions to midterm-1 : Probability theory (2019).

1. (1) True. For any $\delta>0$ $E[|X|^n] \geq (2+\delta)^n P\{|X|\geq 2+\delta\} \Rightarrow P\{|X|\geq 2+\delta\} \leq \frac{n2^n}{(2+\delta)^n} \to 0$ as $n\to\infty$

Hence $P\{|X|\geq 2+\frac{1}{k}\}=0$ $\forall k=1,2,3,\ldots$, take intersection to get

$P\{|X|>2\}=0$

(2) True. $\sigma(X)=\{\vec{X}^{1}(B) \mid B\in\mathcal{B}_{\mathbb{R}}\}$. Hence if $X(s)=X(t)\overset{\alpha}{=}$ for some $s<t$ then $\vec{X}(B)$ contains both s,t if $\alpha\in B$ and contains neither if $\alpha\notin B$. However $\sigma(X)=\cancel{\ }\mathcal{B}_{[0,1]}$ hence $[0,u]\in\sigma(X)$ for any u. Take $s<u<t$ to get a set that contains s but not t. Contradiction. X must be injective.

(3) True. Given $\epsilon>0$ $\exists M$ s.t. $\mu_i([-M,M])\geq 1-\frac{\epsilon}{2}$ $\forall i\in I$. Then

$(\mu_i\otimes\mu_j)(\underbrace{[-M,M]^2}_{\text{compact in } \mathbb{R}^2}) = \mu_i([-M,M])\,\mu_j([-M,M]) \geq (1-\frac{\epsilon}{2})^2 > 1-\epsilon$, $\forall i,j\in I$.

(4) True. $\mu(A)=\int_A X\,d\nu$ $\forall A\in\mathcal{F}$ by the given conditions.

Suppose $\nu\ll\mu$: Take $A=\{X=0\}$. Then $\mu(A)=\int X\mathbb{1}_{X=0}\,d\nu=0$

Hence must have $\nu(A)=0$. ie $X>0$ a.s. $[\nu]$

Suppose $X>0$ a.s. $[\nu]$: If $\mu(A)=0$ then $\int X\mathbb{1}_A\,d\nu=0$. But $X\mathbb{1}_A\geq 0$,

Hence by positivity of integral, $X\mathbb{1}_A=0$ a.s. $[\nu]$. i.e., $\nu\{\{X>0\}\cap A\}=0$

~~$\nu\{\{X=0\}\cap A\}=0$. But $\nu\{X=0\}=0$~~

Since $\nu\{X>0\}=1$, it follows that $\nu(A)=0$.

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2. (1) If $S=\{A_1,\ldots,A_k\}$, form all 2^k intersections $B_1\cap\ldots\cap B_k$ where $B_i=A_i$ or A_i^c. Some of these could be empty, but there are 2^k p.w. disjoint sets - call them C_i's.

Then $\sigma(S)=\sigma(C_1,\ldots,C_{2^k})$ ($\because C_i\in\sigma(S)$ $\forall i$ and $A_i\in\sigma(C_1,\ldots,C_{2^k})$ $\forall i$)

and $\sigma(C_1,\ldots,C_{2^k})$ = set of all finite unions of C_i's. Thus $|\sigma(S)|\leq 2^{2^k}$.

(2) Let $\mathcal{G}=\{A \mid A\in\mathcal{F}$, and given $\epsilon>0$ $\exists B_\epsilon\in\mathcal{A}$ s.t. $\mu(A\Delta B_\epsilon)<\epsilon\}$.

~~By assumption~~ $\mathcal{G}\supseteq\mathcal{A}$ ($\because$ we can take $B=A$ for any ϵ, $\mu(A\Delta A)=0$)

Further, if $A_n\in\mathcal{G}$, $A_n\uparrow A$, then find $B_n\in\mathcal{A}$ s.t. $\mu(A_n\Delta B_n)<\frac{\epsilon}{2^n}$.

Hence if $B=\bigcup_n B_n$, then $\mu(A\Delta B)<\sum\mu(A_n\Delta B_n)<\epsilon$. Further if $C_n=\bigcup_{k=1}^n B_k$

then $\mu(C_n)\uparrow\mu(B)$, hence $\mu(B\Delta C_n)<\epsilon$ for some n. Thus $\mu(A\Delta C_n)<\epsilon$

But $C_n \in \mathcal{A}$ (a finite union of $B_k \cap A$ and $B_k \in \mathcal{A}$).

Similarly we can show that $\mathcal{G}$ is closed under decreasing intersections

(better: check that $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$, hence convert $A_n \downarrow A$ to $A_n^c \uparrow A^c \ldots$).

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3. (1) Let $T = S^c = [0,1] \setminus S$. $T = \bigcup_{b=2}^{\infty} T_b$ where $T_b = \{x \in [0,1] \mid \text{base } b \text{ expansion of } x \text{ misses some digit}\}$

$= \bigcup_{b=2}^{\infty} \bigcup_{k=0}^{b-1} T_{b,k}$ $\quad T_{b,k} = \{x \in [0,1] \mid \text{base } b \text{ expansion of } x \text{ misses digit } k\}$

$= \bigcup_{b=2}^{\infty} \bigcup_{k=0}^{b-1} \bigcap_{n=1}^{\infty} T_{b,k,n}$ $\quad T_{b,k,n} = \{x \in [0,1] \mid \text{base } b \text{ expansion does not have } k \text{ in first } n \text{ digits}\}$

Clearly $\lambda(T_{b,k,n}) = \left(\frac{b-1}{b}\right)^n \to 0$ and $T_{b,k,n}$ are decreasing,

hence $\lambda\left(\bigcap_{n=1}^{\infty} T_{b,k,n}\right) = 0$ $\forall b,k$. Countable union of zero measure sets

has zero measure, hence $\lambda(T) = 0 \Rightarrow \lambda(S) = 1$.

(2) (X,Y) has density $\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$ on $\mathbb{R}^2$.

Let $U = \frac{X}{Y}$, $V = Y$ so $(U,V) = T(X,Y)$ where $T^{-1}(u,v) = (uv, v)$.

$J_{T^{-1}}(u,v) = \det \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} = v$

By change of variable formula, (U,V) has density

$$f_{(U,V)}(u,v) = f_{(X,Y)}(T^{-1}(u,v)) \cdot |J_{T^{-1}}(u,v)|$$

$$= \frac{1}{2\pi} \cdot |v| \, e^{-\frac{1}{2}v^2(1+u^2)}$$

$$\text{Density of } U = \int_{\mathbb{R}} \frac{1}{2\pi} e^{-\frac{1}{2}(1+u^2)v^2} |v| \, dv = \frac{1}{\pi} \int_0^{\infty} e^{-v^2(1+u^2)\frac{1}{2}} v \, dv$$

$$\left(\begin{array}{c}\text{substitute} \\ t = \frac{1}{2}v^2\end{array}\right) = \frac{1}{\pi} \int_0^{\infty} e^{-t(1+u^2)} dt = \frac{1}{\pi(1+u^2)}.$$

Hence $\frac{X}{Y}$ has standard Cauchy distribution.

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(4) (1) $\dfrac{\log x}{x^p + x^{-p}}$ is continuous on $(0,\infty)$. Further $\dfrac{\log x}{x^p+x^{-p}} \le \dfrac{\log x}{x^p} \to 0$ as $x \to \infty$

This implies that the function is bounded $\dfrac{\log x}{x^p+x^{-p}} \le \dfrac{\log x}{x^{-p}} \to 0$ as $x \to 0$
on $(0,\infty)$. That is,
$$|\log x| \le C(x^p + x^{-p}) \text{ for some } C<\infty.$$
Hence if $E[X^p]<\infty$ and $E[X^{-p}]<\infty$, then $E|\log X| \le C(EX^p + EX^{-p}) < \infty$.

(2) Observe that $\displaystyle\sum_{n=1}^{\infty} 1_{n \le X} \le X \le \sum_{n=0}^{\infty} 1_{n\le X}$ $\begin{pmatrix}\text{left sum} = \lfloor X \rfloor \\ \text{right sum} = \lfloor X \rfloor + 1\end{pmatrix}$

By MCT $\displaystyle E\left[\sum_{n=1}^{\infty} 1_{n\le X}\right] = \sum_{n=1}^{\infty} P(X \ge n)$ and $\displaystyle E\left[\sum_{n=0}^{\infty} 1_{n\le X}\right] = \sum_{n=0}^{\infty} P(X\ge n)$

Therefore $E[X]$ is sandwiched between the two sums. Hence
$E[X]<\infty$ if and only if $\sum_n P(X \ge n) < \infty$.

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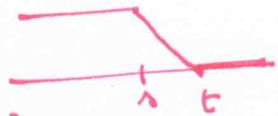
5. (1) For $s<t$ define $\varphi_{s,t}: \mathbb{R} \to [0,1]$ by $\varphi_{s,t}(x) = \begin{cases} 1 & \text{if } x \le s \\ 0 & \text{if } x \ge t \\ \text{linear on } [s,t] \end{cases}$



Then $1_{(-\infty,s]} \le \varphi_{s,t} \le 1_{(-\infty,t]}$. Also $\varphi_{s,t}$ is bounded and continuous.

Hence $\mu_n(-\infty,s] \le \int \varphi_{s,t}\, d\mu_n \xrightarrow{\text{(assumption)}} \int \varphi_{s,t}\, d\mu \le \mu(-\infty,t]$.

Fix s, let ~~t↓s~~ $t \downarrow s$ and use right continuity of F_μ to get
$$\limsup_{n\to\infty} F_{\mu_n}(s) \le F_\mu(s) \quad \forall s. \quad -①$$

Similarly $\mu(-\infty,s] \le \int \varphi_{s,t}\, d\mu = \lim_{n\to\infty} \int \varphi_{s,t}\, d\mu_n \le \liminf_{n\to\infty} \mu_n(-\infty,t]$

Fix t, let $s \uparrow t$ to get $\quad F_\mu(t-) \le \liminf_{n\to\infty} F_{\mu_n}(t). \ \forall t. \ -②$

From ① and ②, if F_μ is continuous at $u \in \mathbb{R}$, then $F_{\mu_n}(u) \to F_\mu(u)$.
Thus $\mu_n \xrightarrow{d} \mu$.

(2) Fix any $A \in \mathcal{B}_{\mathbb{R}}$. Then $\displaystyle\liminf_{n\to\infty} \int_A \varphi_n(x)\,dx \geq \int_A \varphi(x)\,dx$ — ① by Fatou's lemma.

Apply the same to A^c.
$$\liminf_{n\to\infty} \int_{A^c} \varphi_n(x)\,dx \geq \int_{A^c} \varphi(x)\,dx$$

$$\Rightarrow \quad \liminf_{n\to\infty} \left[1 - \int_A \varphi_n(x)\,dx\right] \geq 1 - \int_A \varphi(x)\,dx$$

$$\Rightarrow \quad 1 - \limsup_{n\to\infty} \int_A \varphi_n(x)\,dx \geq 1 - \int_A \varphi(x)\,dx$$

$$\Rightarrow \quad \limsup_{n\to\infty} \int_A \varphi_n(x)\,dx \leq \int_A \varphi(x)\,dx \quad - ②$$

From ① and ②, $\displaystyle\int_A \varphi_n(x)\,dx \to \int_A \varphi(x)\,dx \quad \forall A \in \mathcal{B}_{\mathbb{R}}$.

Take $A = (-\infty, t]$ to get $F_{\mu_n}(t) \to F_\mu(t)$ $\forall t \in \mathbb{R}$. $\Rightarrow \mu_n \xrightarrow{d} \mu$.

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(2) Fix any $A \in \mathcal{B}_{\mathbb{R}}$. Then $\liminf\limits_{n\to\infty} \int_A \varphi_n(x)\,dx \geq \int_A \varphi(x)\,dx$ — ① by Fatou's lemma.

Apply the same to A^c. $\liminf\limits_{n\to\infty} \int_{A^c} \varphi_n(x)\,dx \geq \int_{A^c} \varphi(x)\,dx$

$$\Rightarrow \liminf_{n\to\infty} \left[1 - \int_A \varphi_n(x)\,dx\right] \geq 1 - \int_A \varphi(x)\,dx$$

$$\Rightarrow 1 - \limsup_{n\to\infty} \int_A \varphi_n(x)\,dx \geq 1 - \int_A \varphi(x)\,dx$$

$$\Rightarrow \limsup_{n\to\infty} \int_A \varphi_n(x)\,dx \leq \int_A \varphi(x)\,dx \quad \text{— ②}$$

From ① and ②, $\int_A \varphi_n(x)\,dx \to \int_A \varphi(x)\,dx \quad \forall A \in \mathcal{B}_{\mathbb{R}}$.

Take $A = (-\infty, t]$ to get $F_{\mu_n}(t) \to F_{\mu}(t) \; \forall t \in \mathbb{R}$. $\Rightarrow \mu_n \xrightarrow{d} \mu$.
