MID-TERM EXAMINATION 1
(PROBABILITY THEORY)

28/SEP/2019, 10:00-12:30

ANSWER AS MUCH AS YOU CAN. THE MAXIMUM YOU CAN SCORE IS 50.

Problem 1. (**4×4 marks**) State whether true or false and justify.

(1) If X is a random variable such that $\mathbf{E}[|X|^n] \le n2^n$ for all n, then $|X| \le 2$, $a.s.$

(2) Let $X : [0,1] \mapsto \mathbb{R}$ be Borel measurable. If the sigma-algebra generated by X is the Borel sigma-algebra on $[0,1]$, then X is an injective function.

(3) If $\{\mu_i : i \in I\} \subseteq \mathcal{P}(\mathbb{R})$ is tight, then $\{\mu_i \otimes \mu_j : i,j \in I\}$ is tight in $\mathcal{P}(\mathbb{R}^2)$.

(4) If $\mu \ll \nu$ and X is the density of μ with respect to ν, then $\nu \ll \mu$ if and only if $X > 0$ $a.s.$ $[\nu]$.

Problem 2. (**10 marks**) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

(1) Let $S \subseteq \mathcal{F}$ be a finite collection of sets. Then show that $\sigma(S)$ is also a finite collection.

(2) Let $\mathcal{A}$ be an algebra that generates $\mathcal{F}$. For any $A \in \mathcal{F}$ and any $\epsilon > 0$, show that there is some $B \in \mathcal{A}$ such that $\mu(A \Delta B) < \epsilon$ (here $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference).

Problem 3. (**10 marks**)

(1) Let S be the set of all $x \in [0,1]$ whose base b-expansion contains all the digits $0, 1, \ldots, b-1$, for every $b \in \{2, 3, 4 \ldots\}$. Show that $\lambda(S) = 1$, where λ is the Lebesgue measure on $[0,1]$.

(2) Let X, Y be i.i.d. $N(0,1)$ random variables. Find the density of X/Y.

Problem 4. (**10 marks**) Let $X \ge 0$ be a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) Assume that $\mathbf{E}[X^p] < \infty$ and $\mathbf{E}[X^{-p}] < \infty$ for some $p > 0$. Then show that $\log X$ is integrable.

(2) Show that $\mathbf{E}[X] < \infty$ if and only if $\sum_{n=1}^{\infty} \mathbf{P}\{X \ge n\} < \infty$.

Problem 5. (**10 marks**) Let μ_n, μ be Borel probability measures on $\mathbb{R}$. Show that $\mu_n \overset{d}{\to} \mu$ in each of the following situations (the two parts are not related to each other).

(1) $\int f d\mu_n \to \int f d\mu$ for each bounded continuous function $f : \mathbb{R} \mapsto \mathbb{R}$.

(2) μ_n, μ have densities φ_n, φ with respect to Lebesgue measure on $\mathbb{R}$, and $\varphi_n \to \varphi$ $a.e.$ (w.r.t Lebesgue measure).