HOMEWORK 1

Only submit the coloured problems.

Problem 1. Let Ω be a non-empty set. We say that A_ns increase to A or that A is the increasing limit of A_ns and write $A_n \uparrow A$ if $A_1 \subseteq A_2 \subseteq \dots$ and $A = \cup_n A_n$.

(1) Let $\mathcal{F}$ be a collection of subsets of Ω that is closed under complements and finite unions. Show that $\mathcal{F}$ is a sigma-algebra if and only if it is also closed under increasing limits (i.e., if $A_n \in \mathcal{F}$ and $A_n \uparrow A$, then $A \in \mathcal{F}$).

(2) Let $\mathcal{F}$ be a sigma-algebra and $\mu : \mathcal{F} \to [0, 1]$ be finitely additive (i.e., if $A_1, \dots, A_n \in \mathcal{F}$ are pairwise disjoint, then $\mu(A_1 \cup \dots \cup A_n) = \mu(A_1) + \dots + \mu(A_n)$) and $\mu(\Omega) = 1$. Show that μ is a probability measure if and only if $\mu(A_n) \uparrow \mu(A)$ whenever $\mathcal{F} \ni A_n \uparrow A$.

Problem 2. Show that each of the following collection of subsets of $\mathbb{R}^d$ generate the same sigma-algebra (which we call the Borel sigma-algebra).

(1) $\{(a, b] : a < b\}$.

(2) $\{[a, b] : a \leq b \text{ and } a, b \in \mathbb{Q}\}$.

(3) The collection of all open sets.

(4) The collection of all compact sets.

Problem 3. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measure spaces. If $T : X \to Y$ is a function, show that

(1) $\{T^{-1}B : B \in \mathcal{G}\}$ is a sigma algebra on X and

(2) $\{B \in \mathcal{G} : T^{-1}B \in \mathcal{F}\}$ is sigma-algebra on Y.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that f is measurable (w.r.t. Borel sigma-algebra on the domain and range) under any of the following conditions.

(1) f is increasing ($f(s) \leq f(t)$ if $s \leq t$).

(2) f is continuous.

(3) f is right-continuous.

(4) f is upper semi-continuous (i.e., $f(x) = \limsup_{y \to x} f(y)$ for all x).

Problem 5. Let λ denote the uniform measure on $[0, 1]$.

(1) Find the push-forward of λ under $x \mapsto x^3$, under $x \mapsto -\log x$, under $x \mapsto 1/x$.

(2) Find the maps from $[0, 1]$ to $\mathbb{R}$ so that the push-forward of λ is $\text{Exp}(\theta)$, arcsine measure (density $c/\sqrt{1-x^2}$ on $(-1, 1)$) and Cauchy measure (density $1/\pi(1 + x^2)$).