HOMEWORK 2

Only submit the coloured problems.

Problem 1. Let F be the CDF of a Borel probability measure μ on the line.

(1) Show that F is continuous at x if and only if $\mu(\{x\}) = 0$.

(2) Show that F can have at most countably many discontinuities.

(3) Show that given any countable set $\{x_1, x_2, \ldots\}$ and any number $p_1, p_2, \ldots$ such that $\sum_i p_i \leq 1$, there is a probability measure whose CDF has a jump of magnitude p_i at x_i for each i, and no other discontinuities.

Problem 2. Decide whether the following are true or false and explain why.

(1) If X is independent of itself, X is constant a.s.

(2) If X is independent X^2 then X is a constant a.s.

(3) If $X, Y, X + Y$ are independent, then X and Y are constants a.s.

(4) If X and Y are independent and also $X + Y$ and $X - Y$ are independent, then X and Y must be constants a.s.

Problem 3. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

(1) If X is a d-dimensional random vector and $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable, show that $g \circ X$ is a random variable.

(2) If X, Y are random variables, show that $X + Y$, XY and X^p are all random variables.

Problem 4. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Define

$$\mathcal{G} = \{A \subseteq \Omega : \text{ there exist } B, C \in \mathcal{F} \text{ such that } B \subseteq A \subseteq C \text{ and } \mathbf{P}(B) = \mathbf{P}(C)\}.$$

Define $\mathbf{Q} : \mathcal{G} \to [0, 1]$ by setting $\mathbf{Q}(A) = \mathbf{Q}(B)$ for such A. Show that $\mathcal{G}$ is a sigma-algebra that contains $\mathcal{F}$ and $\mathbf{Q}$ is a probability measure on $\mathcal{G}$ that extends $\mathbf{P}$.

Problem 5. Given a random number generator that can give random numbers from the uniform distribution on $[0, 1]$, explain how you would generate random numbers from the following distributions: Bin(n, p), Pois(λ), Geo(p), Exp(λ), $N(0, 1)$.

Problem 6. Give an example to show that if two probability measures on a sigma-algebra are equal on a generating collection, they do not necessarily have to be equal on the sigma-algebra. But if the generating collection is a π-system, then they are equal.

Problem 7. Let $X_1, \ldots, X_n$ be random variables on a common probability space. Let F be the joint CDF (i.e., $F(x_1, \ldots, x_n) = \mathbf{P}\{X_1 \le x_1, \ldots, X_n \le x_n\}$) and let F_j be the CDF of X_j. Show that X_is are independent if and only if $F(x_1, \ldots, x_n) = F_1(x_1) \ldots F_n(x_n)$ for all $x_1, \ldots, x_n \in \mathbb{R}$.