# PROBLEMS IN PROBABILITY THEORY

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**Problem 1.** Let $\mathcal{F}$ be a $\sigma$-algebra of subsets of $\Omega$.

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), under symmetric differences ($A \Delta B$).

(2) If $A_n$ is a countable sequence of subsets of $\Omega$, the set $\limsup_n A_n$ (respectively $\liminf_n A_n$) is defined as the set of all $\omega \in \Omega$ that belongs to infinitely many (respectively, all but finitely many) of the sets $A_n$.

If $A_n \in \mathcal{F}$ for all $n$, show that $\limsup A_n \in \mathcal{F}$ and $\liminf A_n \in \mathcal{F}$. [**Hint:** First express $\limsup A_n$ and $\liminf A_n$ in terms of $A_n$s and basic set operations].

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, what are $\limsup A_n$ and $\liminf A_n$?

**Problem 2.** Let $(\Omega, \mathcal{F})$ be a set with a $\sigma$-algebra.

(1) Suppose $\mathbf{P}$ is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and $A_n$ increase to $A$ (respectively, decrease to $A$), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P} : \mathcal{F} \to [0,1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) $\mathbf{P}$ is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and $A_n$s increase to $A$, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then, show that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

**Problem 3.** Suppose $S$ is a $\pi$-system and is further closed under complements ($A \in S$ implies $A^c \in S$). Show that $S$ is an algebra.

**Problem 4.** Let $\mathbf{P}$ be a p.m. on a $\sigma$-algebra $\mathcal{F}$ and suppose $S \subseteq \mathcal{F}$ be a $\pi$-system. If $A_k \in S$ for $k \leq n$, write $\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n)$ in terms of probabilities of sets in $S$.

**Problem 5.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbf{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a $\sigma$-algebra.

**Problem 6.** Suppose $\sigma(S) = \mathcal{F}$ and $\mathbf{P}, \mathbf{Q}$ are two probability measure on $\mathcal{F}$. If $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in S$, is it necessarily true that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$? If yes, prove it. If not, give a counterexample.

**Problem 7.** (1) Let $\mathcal{B}$ be the Borel sigma-algebra of $\mathbb{R}$. Show that $\mathcal{B}$ contains all closed sets, all compact sets, all intervals of the form $(a, b]$ and $[a, b)$.

(2) Show that there is a countable family $\mathcal{S}$ of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$.

(3) Let $K$ be the $1/3$-Cantor set. Show that $\mu_*(K) = 0$.

**Problem 8.** Show that each of the following collection of subsets of $\mathbb{R}^d$ generate the same sigma-algebra (which we call the Borel sigma-algebra).

(1) $\{(a, b] : a < b\}$.

(2) $\{[a, b] : a \leq b$ and $a, b \in \mathbf{Q}\}$.

(3) The collection of all open sets.

(4) The collection of all compact sets.

**Problem 9.** (1) Let $X$ be an arbitrary set. Let $S$ be the collection of all singletons in $\Omega$. Describe $\sigma(S)$.

(2) Let $S = \{(a, b] \cup [-b, -a) : a < b$ are real numbers$\}$. Show that $\sigma(S)$ is strictly smaller than the Borel $\sigma$-algebra of $\mathbb{R}$.

(3) Suppose $S$ is a collection of subsets of $X$ and $a, b$ are two elements of $X$ such that any set in $S$ either contains $a$ and $b$ both, or contains neither. Let $\mathcal{F} = \sigma(S)$. Show that any set in $\mathcal{F}$ has the same property (either contains both $a$ and $b$ or contains neither).

**Problem 10.** Let $\Omega$ be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A$ is finite or $A^c$ is finite $\}$. Define $\mu : \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if $A$ is finite and $\mu(A) = 1$ if $A^c$ is finite.

(1) Show that $\mathcal{A}$ is an algebra and that $\mu$ is finitely additive on $\mathcal{A}$.

(2) Under what conditions does $\mu$ extend to a probability measure on $\mathcal{F} = \sigma(\mathcal{A})$?

**Problem 11.** On $\mathbb{N} = \{1, 2, \ldots\}$, let $A_p$ denote the subset of numbers divisible by $p$. Describe $\sigma(\{A_p : p$ is prime$\})$ as explicitly as possible.

**Problem 12.** If $\mathcal{G} \subseteq \mathcal{F}$ are sigma algebras on $\Omega$ and $\mathcal{F}$ is countably generated, then is it necessarily true that $\mathcal{G}$ is countably generated?

**Problem 13.** Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measure spaces. If $T : X \to Y$ is a function, show that

(1) $\{T^{-1}B : B \in \mathcal{G}\}$ is a sigma algebra on $X$ and

(2) $\{B \in \mathcal{G} : T^{-1}B \in \mathcal{F}\}$ is sigma-algebra on $Y$.

**Problem 14.** Let $A_1, A_2, \ldots$ be a finite or countable partition of a non-empty set $\Omega$ (i.e., $A_i$ are pairwise disjoint and their union is $\Omega$). What is the $\sigma$-algebra generated by the collection of subsets $\{A_n\}$? What is the algebra generated by the same collection of subsets?

**Problem 15.** Let $X = [0,1]^{\mathbb{N}}$ be the countable product of copies of $[0,1]$. We define two sigma algebras of subsets of $X$.

(1) Define a metric on $X$ by $d(x,y) = \sum_n |x_n - y_n| 2^{-n}$. Let $\mathcal{B}_X$ be the Borel sigma-algebra of $(X,d)$. [**Note:** For those who know topology, it is better to define $\mathcal{B}_X$ as the Borel sigma algebra for the product topology on $X$. The point is that the metric is flexible. We can take many or other things (but not $d(x,y) = \sup_n |x_n - y_n|$ !!). What matters is only the topology on $X$.]

(2) Let $\mathcal{C}_X$ be the sigma-algebra generated by the collection of all cylinder sets. Recall that cylinder sets are sets of the form $A = U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$ where $U_i$ are Borel subsets of $[0,1]$.

Show that $\mathcal{B}_X = \mathcal{C}_X$.

**Problem 16.** Let $\mu$ be the Lebesgue p.m. on the Cartheodary $\sigma$-algebra $\bar{\mathcal{B}}$ and let $\mu_*$ be the corresponding outer Lebesgue measure defined on all subsets of $[0,1]$. We say that a subset $N \subseteq [0,1]$ is a null set if $\mu_*(N) = 0$. Show that

$$\bar{\mathcal{B}} = \{B \cup N : B \in \mathcal{B} \text{ and } N \text{ is null}\}$$

where $\mathcal{B}$ is the Borel $\sigma$-algebra of $[0,1]$.

[**Note:** The point of this exercise is to show how much larger is the Lebesgue $\sigma$-algebra than the Borel $\sigma$-algebra. The answer is, not much. Up to a null set, every Lebesgue measurable set is a Borel set. However, cardinality-wise, there is a difference. The Lebesgue $\sigma$-algebra is in bijection with $2^{\mathbb{R}}$ while the Borel $\sigma$-algebra is in bijection with $\mathbb{R}$.]

**Problem 17.** Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. Say that a subset $N \subseteq \Omega$ is $\mathbf{P}$-null if there exists $A \in \mathcal{F}$ with $\mathbf{P}(A) = 0$ and such that $N \subseteq A$. Define $\mathcal{G} = \{A \cup N : A \in \mathcal{F} \text{ and } N \text{ is null}\}$.

(1) Show that $\mathcal{G}$ is a $\sigma$-algebra.

(2) For $A \in \mathcal{G}$, write $A = B \cup N$ with $b \in \mathcal{F}$ and a null set $N$, and define $\mathbf{Q}(A) = \mathbf{P}(B)$. Show that $\mathbf{Q}$ is well-defined, that $\mathbf{Q}$ is a probability measure on $\mathcal{G}$ and $\mathbf{Q}\Big|_{\mathcal{F}} = \mathbf{P}$.

[**Note:** $\mathcal{G}$ is called the $\mathbf{P}$-completion of $\mathcal{F}$. It is a cheap way to enlarge the $\sigma$-algebra and extend the measure to the larger $\sigma$-algebra. Another description of the extended $\sigma$-algebra is $\mathcal{G} = \{A \subseteq \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subseteq A \subseteq C \text{ and } \mathbf{P}(B) = \mathbf{P}(C)\}$. Combined with the previous problem, we see that the Lebesgue $\sigma$-algebra is just the completion of the Borel $\sigma$-algebra under the Lebesgue measure. However, note that completion depends on the probability measure (for a discrete probability measure on $\mathbb{R}$, the completion will be the power set $\sigma$-algebra!). For this reason, we prefer to stick to the Borel $\sigma$-algebra and not bother to extend it.]

**Problem 18.** Follow these steps to obtain Sierpinski's construction of a non-measurable set. Here $\mu_*$ is the outer Lebesgue measure on $\mathbb{R}$.

(1) Regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and choose a basis $H$ (why is it possible?).

(2) Let $A_0 = H \cup (-H) = \{x : x \in H \text{ or } -x \in H\}$. For $n \geq 1$, define $A_n := A_{n-1} - A_{n-1}$ (may also write $A_n = A_{n-1} + A_{n-1}$ since $A_0$ is symmetric about 0). Show that $\bigcup_{n\geq 0} \bigcup_{q \geq 1} \frac{1}{q} A_n = \mathbb{R}$ where $\frac{1}{q}A_n$ is the set $\{\frac{x}{q} : x \in A_n\}$.

(3) Let $N := \min\{n \geq 0 : \mu_*(A_n) > 0\}$ (you must show that $N$ is finite!). If $A_N$ is measurable, show that $\cup_{n \geq N+1} A_n = \mathbb{R}$.

(4) Get a contradiction to the fact that $H$ is a basis and conclude that $A_N$ cannot be measurable.

[**Remark:** If you start with $H$ which has zero Lebesgue measure, then $N \geq 1$ and $A := E_{N-1}$ is a Lebesgue measurable set such that $A + A$ is not Lebesgue measurable! That was the motivation for Sierpinski. To find such a basis $H$, show that the Cantor set spans $\mathbb{R}$ and then choose a basis $H$ contained inside the Cantor set.]

**Problem 19.** We saw that for a Borel probability measure $\mu$ on $\mathbb{R}$, the pushforward of Lebesgue measure on $[0, 1]$ under the map $F_\mu^{-1} : [0, 1] \to \mathbb{R}$ (as defined in lectures) is precisely $\mu$. This is also a practical tool in simulating random variables. We assume that a random number generator gives us uniform random numbers from $[0, 1]$. Apply the above idea to simulate random numbers from the following distributions (in matlab/mathematica or a program of your choice) a large number of times and compare the histogram to the actual density/mass function.

(1) Uniform distribution on $[a, b]$, (2) Exponential($\lambda$) distribution, (3) Cauchy distribution, (4) Poisson($\lambda$) distribution. What about the normal distribution?

**Problem 20.** Let $\Omega = X = \mathbb{R}$ and let $T : \Omega \to X$ be defined by $T(x) = x$. We give a pair of $\sigma$-algebras, $\mathcal{F}$ on $\Omega$ and $\mathcal{G}$ on $X$ by taking $\mathcal{F}$ and $\mathcal{G}$ to be one of $2^{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{R}}$ or $\{\emptyset, \mathbb{R}\}$. Decide for each of the nine pairs, whether $T$ is measurable or not.

**Problem 21.**

(1) Define $T : \Omega \to \mathbb{R}^n$ by $T(\omega) = (\mathbf{1}_{A_1}(\omega), \ldots, \mathbf{1}_{A_n}(\omega))$ where $A_1, \ldots, A_n$ are given subsets of $\Omega$. What is the smallest $\sigma$-algebra on $\Omega$ for which $T$ becomes a random variable?

(2) Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and assume that $A_k \in \mathcal{F}$. Describe the push-forward measure $\mathbf{P} \circ T^{-1}$ on $\mathbb{R}^n$.

**Problem 22.** For $k \geq 0$, define the functions $r_k : [0,1) \to \mathbb{R}$ by writing $[0,1) = \bigsqcup_{0 \leq j < 2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k}, (j+1)2^{-k})$ and setting

$$r_k(x) = \begin{cases} -1 & \text{if } x \in I_j^{(k)} \text{ for odd } j, \\ +1 & \text{if } x \in I_j^{(k)} \text{ for even } j. \end{cases}$$

Fix $n \geq 1$ and define $T_n : [0,1) \to \{-1,1\}^n$ by $T_n(x) = (r_0(x), \ldots, r_{n-1}(x))$. Find the push-forward of the Lebesgue measure on $[0,1)$ under $T_n$

**Problem 23.** Let $\mathcal{G}$ be the countable-cocountable sigma algebra on $\mathbb{R}$. Define the probability measure $\mu$ on $\mathcal{G}$ by $\mu(A) = 0$ if $A$ is countable and $\mu(A) = 1$ if $A^c$ is countable. Show that $\mu$ is *not* the push-forward of Lebesgue measure on $[0,1]$, i.e., there does not exist a measurable function $T : [0,1] \mapsto \Omega$ (w.r.t. the $\sigma$-algebras $\mathcal{B}$ and $\mathcal{G}$) such that $\mu = \lambda \circ T^{-1}$.

**Problem 24.** (1) If $T : \mathbb{R}^n \to \mathbb{R}^m$, show that $T$ is Borel measurable if it is (a) continuous or (b) right continuous or (c) lower semicontinuous or (d) non-decreasing (take $m = n = 1$ for the last one).

(2) If $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with the Lebesgue sigma-algebra, show that even if $T$ is continuous, it need not be measurable! Just do this for $n = m = 1$.

**Problem 25.** Show that composition of random variables is a random variable. Show that real-valued random variables on a given $(\Omega, \mathcal{F})$ are closed under linear combinations, under multiplication, under countable suprema (or infima) and under limsup (or liminf) of countable sequences.

**Problem 26.** Let $\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{k/n}$ and let $\mu$ be the uniform p.m. on $[0,1]$. Show directly by definition that $d(\mu_n, \mu) \to 0$ as $n \to \infty$.

**Problem 27 (Change of variable for densities).** (1) Let $\mu$ be a p.m. on $\mathbb{R}$ with density $f$ by which we mean that its CDF $F_\mu(x) = \int_{-\infty}^{x} f(t)dt$ (you may assume that $f$ is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}} f = 1$). Then, find the (density of the) push forward measure of $\mu$ under (a) $T(x) = x + a$ (b) $T(x) = bx$ (c) $T$ is any increasing and differentiable function.

(2) If $X$ has $N(\mu, \sigma^2)$ distribution, find the distribution of $(X - \mu)/\sigma$.

**Problem 28.** (1) Let $X = (X_1, \ldots, X_n)$. Show that $X$ is an $\mathbb{R}^d$-valued r.v. if and only if $X_1, \ldots, X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1), \ldots, \sigma(X_n)$?

(2) Let $X : \Omega_1 \to \Omega_2$ be a random variable. If $X(\omega) = X(\omega')$ for some $\omega, \omega' \in \Omega_1$, show that there is no set $A \in \sigma(X)$ such that $\omega \in A$ and $\omega' \notin A$ or vice versa. **[Extra!** If $Y : \Omega_1 \to \Omega_2$ is another r.v. which is measurable w.r.t. $\sigma(X)$ on $\Omega_1$, then show that $Y$ is a function of $X$].

**Problem 29 (Lévy metric).** (1) Show that the Lévy metric on $\mathcal{P}(\mathbb{R}^d)$ defined in class is actually a metric.

(2) Show that under the Lévy metric, $\mathcal{P}(\mathbb{R}^d)$ is a complete and seperable metric space.

**Problem 30 (Lévy-Prohorov metric).** If $(X, d)$ is a metric space, let $\mathcal{P}(X)$ denote the space of Borel probability measures on $X$. For $\mu, \nu \in \mathcal{P}(X)$, define

$$D(\mu, \nu) = \inf\{r \geq 0 : \mu(A_r) + r \geq \nu(A) \text{ and } \nu(A_r) + r \geq \mu(A) \text{ for all closed sets } A\}.$$

Here $A_r = \{y \in X : d(x, y) \leq r \text{ for some } x \in A\}$ is the closed $r$-neighbourhood of $A$.

(1) Show that $D$ is a metric on $\mathcal{P}(X)$.

(2) When $X$ is $\mathbb{R}^d$, show that this agrees with the definition of Lévy metric given in class (i.e., for any $\mu_n, \mu$, we have that $\mu_n \to \mu$ in both metrics or neither).

**Problem 31 (Lévy metric).** Let $\mathcal{P}([-1, 1]) \subseteq \mathcal{P}(\mathbb{R})$ be the set of all Borel probability measures $\mu$ such that $\mu([-1, 1]) = 1$. For $\epsilon > 0$, find a finite $\epsilon$-net for $\mathcal{P}([-1, 1])$. [*Note:* Recall that an $\epsilon$-net means a subset such that every element of $\mathcal{P}([-1, 1])$ is within $\epsilon$ distance of some element of the subset. Since $\mathcal{P}([-1, 1])$ is compact, we know that a finite $\epsilon$-net exists for all $\epsilon > 0$.]

**Problem 32.** On the probabiity space $([0, 1], \mathcal{B}, \mu)$, for $k \geq 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \geq 1$, what is the distribution of $X_n$?

(2) For any fixed $n \geq 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

[**Note:** $X_k(t)$ is just the $k^{\text{th}}$ digit in the binary expansion of $t$. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t = 1$)].

**Problem 33 (Coin tossing space).** Continuing with the previous example, consider the mapping $X : [0, 1] \to \{0, 1\}^{\mathbb{N}}$ defined by $X(t) = (X_1(t), X_2(t), \ldots)$. With the Borel $\sigma$-algebra on $[0, 1]$ and the $\sigma$-algebra generated by cylinder sets on $\{0, 1\}^{\mathbb{N}}$, show that $X$ is a random variable and find the push-foward of the Lebesue measure under $X$.

**Problem 34 (Equivalent conditions for weak convergence).** Show that the following statements are equivalent to $\mu_n \xrightarrow{d} \mu$ (you may work in $\mathcal{P}(\mathbb{R})$).

(1) $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ if $F$ is closed.

(2) $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ if $G$ is open.

(3) $\limsup_{n\to\infty} \mu_n(A) = \mu(A)$ if $A \in \mathcal{F}$ and $\mu(\partial A) = 0$.

**Problem 35.** Fix $\mu \in \mathcal{P}(\mathbb{R})$. For $s \in \mathbb{R}$ and $r > 0$, let $\mu_{r,s} \in \mathcal{P}(\mathbb{R})$ be defined as $\mu_{r,s}(A) = \mu(rA + s)$ where $rA+s = \{rx+s : x \in A\}$. For which $R \subseteq (0,\infty)$ and $S \subseteq \mathbb{R}$ is it true that $\{\mu_{r,s} : r \in R,\ s \in S\}$ a tight family? [*Remark:* If not clear, just take $\mu$ to be the Lebesgue measure on $[0,1]$.]

**Problem 36.** (1) Show that the family of Normal distributions $\{N(\mu,\sigma^2) : \mu \in \mathbb{R} \text{ and } \sigma^2 > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ and $B \subseteq (0,\infty)$ is the restricted family $\{N(\mu,\sigma^2) : \mu \in A \text{ and } \sigma^2 \in B\}$ tight?

**Problem 37.** (1) Show that the family of exponential distributions $\{\text{Exp}(\lambda) : \lambda > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ is the restricted family $\{\text{Exp}(\lambda) : \lambda > 0\}$ tight?

**Problem 38.** Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and that the distribution function of $\mu$ is continuous. If $\mu_n \xrightarrow{d} \mu$, show that $F_{\mu_n}(t) - F_\mu(t) \to 0$ uniformly over $t \in \mathbb{R}$. [*Restatement:* When $F_\mu$ is continuous, convergence to $\mu$ in Lévy-Prohorov metric also implies convergence in Kolmogorov-Smirnov metric. ]

**Problem 39.** Show that the statement in the previous problem cannot be quantified. That is,

Given any $\epsilon_n \downarrow 0$ (however fast) and $\delta_n \downarrow 0$ (however slow), show that there is some $\mu_n, \mu$ with $F_\mu$ continuous, such that $d_{LP}(\mu_n,\mu) \leq \epsilon_n$ and $d_{KS}(\mu_n,\mu) \geq \delta_n$.

**Problem 40.** Consider the family of Normal distributions, $\{N(\mu,\sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$. Show that the map $(\mu,\sigma^2) \to N(\mu,\sigma^2)$ from $\mathbb{R} \times \mathbb{R}_+$ to $\mathcal{P}(\mathbb{R})$ is continuous. (Complicated way of saying that if $(\mu_n,\sigma_n^2) \to (\mu,\sigma^2)$, then $N(\mu_n,\sigma_n^2) \xrightarrow{d} N(\mu,\sigma^2)$).

Do the same for other natural families if distributions, (1) Exp$(\lambda)$, (2) Uniform$[a,b]$, (3) Bin$(n,p)$ (fix $n$ and show continuity in $p$), (4) Pois$(\lambda)$.

**Problem 41.** Suppose $\mu_n, \mu$ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions $(p_n(k))_{k\in\mathbb{Z}}$ and $(p(k))_{k\in\mathbb{Z}}$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

**Problem 42.** Given a Borel p.m. $\mu$ on $\mathbb{R}$, show that it can be written as a convex combination $\alpha\mu_1 + (1-\alpha)\mu_2$ with $\alpha \in [0,1]$, where $\mu_1$ is a purely atomic Borel p.m and $\mu_2$ is a Borel p.m with no atoms.

**Problem 43.** Let $F$ be the CDF of a Borel probability measure $\mu$ on the line.

(1) Show that $F$ is continuous at $x$ if and only if $\mu(\{x\}) = 0$.

(2) Show that $F$ can have at most countably many discontinuities.

(3) Show that given any countable set $\{x_1, x_2, \ldots\}$ and any number $p_1, p_2, \ldots$ such that $\sum_i p_i \leq 1$, there is a probability measure whose CDF has a jump of magnitude $p_i$ at $x_i$ for each $i$, and no other discontinuities.

**Problem 44.** Let $X$ be a random variable with distribution $\mu$ and $X_n$ are random variables defined as follows. If $\mu_n$ is the distribution of $X_n$, in each case, show that $\mu_n \xrightarrow{d} \mu$ as $n \to \infty$.

(1) (Truncation). $X_n = (X \wedge n) \vee (-n)$.

(2) (Discretization). $X_n = \frac{1}{n}\lfloor nX \rfloor$.

**Problem 45.** Consider the space $X = [0,1]^{\mathbb{N}} := \{\mathbf{x} = (x(1), x(2), \ldots) : 0 \leq x(i) \leq 1 \text{ for each } i \in \mathbb{N}\}$. Define the metric $d(\mathbf{x}, \mathbf{y}) = \sup_i \frac{|x(i)-y(i)|}{i}$.

(1) Show that $\mathbf{x}_n \to \mathbf{x}$ in $(X, d)$ if and only if $x_n(i) \to x(i)$ for each $i$, as $n \to \infty$.

[**Note:** What matters is this pointwise convergence criterion, not the specific metric. The resulting topology is called *product topology*. The same convergence would hold if we had defined the metric as $d(\mathbf{x}, \mathbf{y}) = \sum_i 2^{-i}|x(i) - y(i)|$ or $d(\mathbf{x}, \mathbf{y}) = \sum_i i^{-2}|x(i) - y(i)|$ etc., But not the metric $\sup_i |x(i) - y(i)|$ as convergence in this metric is equivalent to uniform convergence over all $i \in \mathbb{N}$].

(2) Show that $X$ is compact.

[**Note:** What is this problem doing here? The purpose is to reiterate a key technique we used in the proof of Helly's selection principle!]

**Problem 46.** Recall the Cantor set $C = \bigcap_n K_n$ where $K_0 = [0,1]$, $K_1 = [0,1/3] \cup [2/3,1]$, etc. In general, $K_n = \bigcup_{1\leq j\leq 2^n} [a_{n,j}, b_{n,j}]$ where $b_{n,j} - a_{n,j} = 3^{-n}$ for each $j$.

(1) Let $\mu_n$ be the uniform probability measure on $K_n$. Describe its CDF $F_n$.

(2) Show that $F_n$ converges uniformly to a CDF $F$.

(3) Let $\mu$ be the probability measure with CDF equal to $F$. Show that $\mu(C) = 1$.

**Problem 47.** Let $\mu \in \mathcal{P}(\mathbb{R})$.

(1) For any $n \geq 1$, define a new probability measure by $\mu_n(A) = \mu(n.A)$ where $n.A = \{nx : x \in A\}$. Does $\mu_n$ converge as $n \to \infty$?

(2) Let $\mu_n$ be defined by its CDF

$$F_n(t) = \begin{cases} 0 & \text{if } t < -n, \\ F(t) \text{ if } -n \leq t < n, & \\ 1 & \text{if } t \geq n. \end{cases}$$

Does $\mu_n$ converge as $n \to \infty$?

(3) In each of the cases, describe $\mu_n$ in terms of random variables. That is, if $X$ has distribution $\mu$, describe a transformation $T_n(X)$ that has the distribution $\mu_n$.

**Problem 48.** (Bernoulli convolutions) For any $\lambda > 1$, define $X_\lambda : [0,1] \to \mathbb{R}$ by $X(\omega) = \sum_{k=1}^{\infty} \lambda^{-k} X_k(\omega)$. Check that $X_\lambda$ is measurable, and define $\mu_\lambda = \mu X_\lambda^{-1}$. Show that for any $\lambda > 2$, show that $\mu_\lambda$ is singular w.r.t. Lebesgue measure.

**Problem 49.** For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of $X$ under the above equivalence relation).

**Problem 50.** (1) Give an example of a sequence of r.v.s $X_n$ such that $\liminf \mathbf{E}[X_n] < \mathbf{E}[\liminf X_n]$.

(2) Give an example of a sequence of r.v.s $X_n$ such that $X_n \overset{a.s.}{\to} X$, $\mathbf{E}[X_n] = 1$, but $\mathbf{E}[X] = 0$.

**Problem 51.** (**Alternate construction of Cantor measure**) Let $K_1 = [0, 1/3] \cup [2/3, 1]$, $K_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$, etc., be the decreasing sequence of compact sets whose intersection is $K$. Observe that $K_n$ is a union of $2^n$ intervals each of length $3^{-n}$. Let $\mu_n$ be the p.m. which is the "renormalized Lebesgue measure" on $K_n$. That is, $\mu_n(A) := 3^n 2^{-n} \mu(A \cap K_n)$ for $A \in \mathcal{B}_\mathbb{R}$. Then each $\mu_n$ is a Borel p.m. Show that $\mu_n \overset{d}{\to} \mu$, the Cantor measure (which was defined differently in class).

**Problem 52.** (**A quantitative characterization of absolute continuity**) Suppose $\mu \ll \nu$. Then, show that given any $\epsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \epsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

**Problem 53.** Suppose $f : [a, b] \to \mathbb{R}$ is a Borel measurable function. Then, show that $g(x) := \int_0^x f(u)du$ is a continuous function on $[0, 1]$. [**Note:** It is in fact true that $g$ is differentiable at almost every $x$ and that $g' = f$ *a.s.*, but that is a more sophisticated fact, called *Lebesgue's differentiation theorem*. In this course, we only need Lebesgue integration, not differentiation. The latter may be covered in your measure theory class].

**Problem 54.** (**Differentiating under the integral**). Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, satisfy the following assumptions.

(1) $x \to f(x, \theta)$ is Borel measurable for each $\theta$.

(2) $\theta \to f(x, \theta)$ is continuously differentiable for each $x$.

(3) $f(x, \theta)$ and $\frac{\partial f}{\partial \theta}(x, \theta)$ are uniformly bounded functions of $(x, \theta)$.

Then, justify the following "differentiation under integral sign" (including the fact that the integrals here make sense).

$$\frac{d}{d\theta} \int_a^b f(x, \theta)dx = \int_a^b \frac{\partial f}{\partial \theta}(x, \theta)\, dx$$

[**Hint:** Remember that derivative is the limit of difference quotients, $h'(t) = \lim_{\epsilon \to 0} \frac{h(t+\epsilon)-h(t)}{\epsilon}$.

**Problem 55.** (1) Let $X \geq 0$ be a r.v on $(\Omega, \mathcal{F}, \mathbf{P})$ with $0 < \mathbf{E}[X] < \infty$. Then, define $\mathbf{Q}(A) = \mathbf{E}[X\mathbf{1}_A]/\mathbf{E}[X]$ for any $A \in \mathcal{F}$. Show that $\mathbf{Q}$ is a probability measure on $\mathcal{F}$. Further, show that for any bounded random variable $Y$, we have $\mathbf{E}_{\mathbf{Q}}[Y] = \frac{\mathbf{E}[YX]}{\mathbf{E}[X]}$.

(2) If $\mu$ and $\nu$ are Borel probability measures on the line with continuous densities $f$ and $g$ (respectively) w.r.t. Lebesgue measure. Under what conditions can you assert that $\mu$ has a density w.r.t $\nu$? In that case, what is that density?

**Problem 56.** For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of $X$ under the equivalence relation $X \sim Y$ if $\mathbf{P}(X = Y) = 1$).

**Problem 57.** Let $X$ be a non-negative random variable.

(1) Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\}dt$ (in particular, if $X$ is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$).

(2) Show that $\mathbf{E}[X^p] = \int_0^\infty pt^{p-1}\mathbf{P}\{X \geq t\}dt$ for any $p > 0$.

**Problem 58.** Let $X$ be a non-negative random variable. If $\mathbf{E}[X]$ is finite, show that $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ is finite for any $a > 0$. Conversely, if $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ for some $a > 0$, show that $\mathbf{E}[X]$ is finite.

**Problem 59.** Show that the values $\mathbf{E}[f \circ X]$ as $f$ varies over the class of all smooth (infinitely differentiable), compactly supported functions determine the distribution of $X$.

**Problem 60.** (i) Express the mean and variance of of $aX + b$ in terms of the same quantities for $X$ ($a, b$ are constants).

(ii) Show that $\text{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$.

**Problem 61.** Compute mean, variance and moments (as many as possible!) of the Normal(0,1), exponential(1), Beta(p,q) distributions.

**Problem 62.**
(1) Suppose $X_n \geq 0$ and $X_n \to X$ a.s. If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, show that $\mathbf{E}[|X_n - X|] \to 0$.

(2) If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{|X|>A}] \to 0$ as $A \to \infty$.

**Problem 63.**
(1) Suppose $(X, Y)$ has a continuous density $f(x, y)$. Find the density of $X/Y$. Apply to the case when $(X, Y)$ has the *standard bivariate normal distribution* with density $f(x, y) = (2\pi)^{-1} \exp\{-\frac{x^2+y^2}{2}\}$.

(2) Find the distribution of $X + Y$ if $(X, Y)$ has the standard bivariate normal distribution.

(3) Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find the density of $(U, V)$.

**Problem 64.** Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for every $f \in C_b(\mathbb{R})$. What if we only assume $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^n)$ - can we conclude that $\mu_n \xrightarrow{d} \mu$?

**Problem 65.** Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$ having densities $f_n, f$ with respect to Lebesgue measure. If $f_n \to f$ a.e. (w.r.t. Lebesgue measure), show that $\mu_n \xrightarrow{d} \mu$.

**Problem 66 (Moment matrices).** Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\alpha_k = \int x^k d\mu(x)$ (assume that all moments exist). Then, for any $n \geq 1$, show that the matrix $(\alpha_{i+j})_{0 \leq i,j \leq n}$ is non-negative definite. **[Suggestion:** First solve $n = 1$].

**Problem 67.** Let $X$ be a non-negative random variable with all moments (i.e., $\mathbf{E}[X^p] < \infty$ for all $p < \infty$). Show that $\log \mathbf{E}[X^p]$ is a convex function of $p$.

**Problem 68.** (1) Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^d)$. Assume that $\mu_n$ has density $f_n$ and $\mu$ has density $f$ w.r.t Lebesgue measure on $\mathbb{R}^n$. If $f_n(t) \to f(t)$ for all $t$, then show that $\mu_n \xrightarrow{d} \mu$.

(2) Show that $N(\mu_n, \sigma_n^2) \xrightarrow{d} N(\mu, \sigma)$ if and only if $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$.

**Problem 69.** (1) Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\alpha', 1)$ be independent random variables on a common probability space. Find the distribution of $\frac{X}{X+Y}$.

(2) If $U, V$ are independent and have uniform([0,1]) distribution, find the distribution of $U+V$.

**Problem 70.** Let $\Omega = \{1, 2, \ldots, n\}$. For a probability measure $\mathbf{P}$ on $\Omega$, we define it "entropy" $H(\mathbf{P}) := -\sum_{k=1}^n p_k \log p_k$ where $p_k = \mathbf{P}\{k\}$ and it is understood that $x \log x = 0$ if $x = 0$. Show that among all probability measures on $\Omega$, the uniform probability measure (the one with $p_k = \frac{1}{n}$ for each $k$) is the unique maximizer of entropy.

**Problem 71.** (1) If $\mu_n \ll \nu$ for each $n$ and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \ll \nu$? If $\mu_n \perp \nu$ for each $n$ and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \perp \nu$? In either case, justify or give a counterexample.

(2) Suppose $X, Y$ are independent (real-valued) random variables with distribution $\mu$ and $\nu$ respectively. If $\mu$ and $\nu$ are absolutely continuous w.r.t Lebesgue measure, show that the distribution of $X + Y$ is also absolutely continuous w.r.t Lebesgue measure.

**Problem 72.** Suppose $\{\mu_\alpha : \alpha \in I\}$ and $\{\nu_\beta : \alpha \in J\}$ are two families of Borel probability measures on $\mathbb{R}$. If both these families are tight, show that the family $\{\mu_\alpha \otimes \nu_\beta : \alpha \in I,\ \beta \in J\}$ is also tight.

**Problem 73.** Let $X$ be a non-negative random variable. If $\mathbf{E}[X] \leq 1$, then show that $\mathbf{E}[X^{-1}] \geq 1$.

**Problem 74.** Suppose $X, Y$ are independent random variables and $X + Y$ has finite expectation. Then show that $X$ has finite expectation. [*Hint:* Assume that $Y$ has symmetric distribution to get a possibly simpler version of the problem]

**Problem 75.** On the probabiity space $([0,1], \mathcal{B}, \mu)$, for $k \geq 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \geq 1$, what is the distribution of $X_n$?

(2) For any fixed $n \geq 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

[**Note:** $X_k(t)$ is just the $k^{\text{th}}$ digit in the binary expansion of $t$. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t = 1$)].

**Problem 76.** If $A \in \mathcal{B}(\mathbb{R}^2)$ has positive Lebesgue measure, show that for some $x \in \mathbb{R}$ the set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$ has positive Lebesgue measure in $\mathbb{R}$.

**Problem 77 (A quantitative characterization of absolute continuity).** Suppose $\mu \ll \nu$. Then, show that given any $\epsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \epsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

**Problem 78.** Let $Z_1, \ldots, Z_n$ be i.i.d $N(0, 1)$ and write $\mathbf{Z}$ for the vector with components $Z_1, \ldots, Z_n$. Let $A$ be an $m \times n$ matrix and let $\mu$ be a vector in $\mathbb{R}^m$. Then the $m$-dimensional random vector $\mathbf{X} = \mu + A\mathbf{Z}$ is said to have distribution $N_m(\mu, \Sigma)$ where $\Sigma = AA^t$ ('Normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$').

(1) If $m \leq n$ and $A$ has rank $m$, show that $\mathbf{X}$ has density $(2\pi)^{-\frac{m}{2}} \exp\{-\frac{1}{2}\mathbf{x}^t A^{-1}\mathbf{x}\}$ w.r.t Lebesgue measure on $\mathbb{R}^m$. In particular, note that the distribution depends only on $\mu$ and $AA^t$. ( **Note:** If $m > n$ or if rank$(A) < m$, then satisfy yourself that $\mathbf{X}$ has no density w.r.t Lebesgue measure on $\mathbb{R}^m$ - you do not need to submit this).

(2) Check that $\mathbf{E}[X_i] = \mu_i$ and $\text{Cov}(X_i, X_j) = \Sigma_{i,j}$.

(3) What is the distribution of (i) $(X_1, \ldots, X_k)$, for $k \leq n$? (ii) $B\mathbf{X}$, where $B$ is a $p \times m$ matrix? (iii) $X_1 + \ldots + X_m$?

**Problem 79.** (1) If $X, Y$ are independent random variables, show that $\text{Cov}(X, Y) = 0$.

(2) Give a counterexample to the converse by giving an infinite sequence of random variables $X_1, X_2, \ldots$ such that $\text{Cov}(X_i, X_j) = 0$ for any $i \neq j$ but such that $X_i$ are not independent.

(3) Suppose $(X_1, \ldots, X_m)$ has (joint) normal distribution (see the first question). If $\text{Cov}(X_i, X_j) = 0$ for all $i \leq k$ and for all $j \geq k + 1$, then show that $(X_1, \ldots, X_k)$ is independent of $(X_{k+1}, \ldots, X_m)$.

**Problem 80.** Decide whether the following are true or false and explain why.

(1) If $X$ is independent of itself, $X$ is constant a.s.

(2) If $X$ is independent $X^2$ then $X$ is a constant a.s.

(3) If $X, Y, X + Y$ are independent, then $X$ and $Y$ are constants a.s.

(4) If $X$ and $Y$ are independent and also $X+Y$ and $X-Y$ are independent, then $X$ and $Y$ must be constants a.s.

**Problem 81.** (1) Suppose $2 \leq k < n$. Give an example of random variables $X_1, \ldots, X_n$ such that any subset of $k$ of these random variables are independent but no subset of $k+1$ of them is independent.

(2) Suppose $(X_1, \ldots, X_n)$ has a multivariate Normal distribution. Show that if $X_i$ are pairwise independent, then they are independent.

**Problem 82.** Show that it is not possible to define uncountably many independent $\text{Ber}(1/2)$ random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

**Problem 83.** Let $X_i$, $i \geq 1$ be random variables on a common probability space. Let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a measurable function (with product sigma algebra on $\mathbb{R}^{\mathbb{N}}$ and Borel sigma algebra on $\mathbb{R}$) and let $Y = f(X_1, X_2, \ldots)$. Show that the distribution of $Y$ depends only on the joint distribution of $(X_1, X_2, \ldots)$ and not on the original probability space. [**Hint:** We used this to say that if $X_i$ are independent Bernoulli random variables, then $\sum_{i \geq 1} X_i 2^{-i}$ has uniform distribution on $[0,1]$, irrespective of the underlying probability space.]

**Problem 84.** Suppose $(X_1, \ldots, X_n)$ has density $f$ (w.r.t Lebesgue measure on $\mathbb{R}^n$).

(1) If $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable functions $g_k$, $k \leq n$. Then show that $X_1, \ldots, X_n$ are independent. (Don't assume that $g_k$ is a density!)

(2) If $X_1, \ldots, X_n$ are independent, then $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable densities $g_1, \ldots, g_n$.

**Problem 85.** (1) Let $S$ be the set of all $x \in [0,1]$ whose base $b$-expansion contains all the digits $0, 1, \ldots, b-1$, for every $b \in \{2, 3, 4 \ldots\}$. Show that $\lambda(S) = 1$, where $\lambda$ is the Lebesgue measure on $[0,1]$.

(2) Let $S$ be the set of all points in $\mathbb{R}^2$ that can be written as a convex combination of two rational points (a rational point is one whose co-ordinates are all rational numbers). Show that $S$ has zero Lebesgue measure.

**Problem 86.** Among all $n!$ permutations of $[n]$, pick one at random with uniform probability. Show that the probability that this random permutation has no fixed points is at most $\frac{1}{2}$ for any $n$.

**Problem 87.** Suppose each of $r = \lambda n$ balls are put into $n$ boxes at random (more than one ball can go into a box). If $N_n$ denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left( \left| \frac{N_n}{n} - e^{-\lambda} \right| > \delta \right) \to 0$$

**Problem 88.** Let $X_n$ be i.i.d random variables such that $\mathbf{E}[|X_1|] < \infty$. Define the random power series $f(z) = \sum_{k=0}^{\infty} X_n z^n$. Show that almost surely, the radius of convergence of $f$ is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_n z^n$ is given by $(\limsup |c_n|^{\frac{1}{n}})^{-1}$].

**Problem 89.**
(1) Let $X$ be a real values random variable with finite variance. Show that $f(a) := \mathbf{E}[(X-a)^2]$ is minimized at $a = \mathbf{E}[X]$.
(2) What is the quantity that minimizes $g(a) = \mathbf{E}[|X-a|]$? [**Hint:** First consider $X$ that takes finitely many values with equal probability each].

**Problem 90** (**Existence of Markov chains**). Let $S$ be a countable set (with the power set sigma algebra). Two ingredients are given: *A transition matrix*, that is, a function $p : S \times S \to [0,1]$ be a function such that $p(x, \cdot)$ is a probability mass function on $S$ for each $x \in S$. (1) *An initial distribution*, that is a probability mass function $\mu_0$ on $S$.

For $n \geq 0$ define the probability measure $\nu_n$ on $S^{n+1}$ (with the product sigma algebra) by

$$\nu_n(A_0 \times A_1 \times \ldots \times A_n) = \sum_{(x_0,\ldots,x_n) \in A_0 \times \ldots \times A_n} \mu_0(x_0) \prod_{j=0}^{n-1} p(x_j, x_{j+1}).$$

Show that $\nu_n$ form a consistent family of probability distributions and conclude that a Markov chain with initial distribution $\mu_0$ and transition matrix $p$ exists.

**Problem 91.** Show that it is not possible to define uncountably many independent $\mathrm{Ber}(1/2)$ random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

**Problem 92.** Let $(\Omega_i, \mathcal{F}_i, \mathbf{P}_i)$, $i \in I$, be probability spaces and let $\Omega = \times_i \Omega_i$ with $\mathcal{F} = \otimes_i \mathcal{F}_i$ and $\mathbf{P} = \otimes_i \mathbf{P}_i$. If $A \in \mathcal{F}$, show that for any $\epsilon > 0$, there is a cylinder set $B$ such that $\mathbf{P}(A \Delta B) < \epsilon$.

**Problem 93.** Let $\xi, \xi_n$ be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that $\limsup_{n \to \infty} |\xi_n|^{\frac{1}{n}} = 1$ *a.s.*

(2) Let $c_n$ be (non-random) complex numbers. Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} c_n \xi_n z^n$ is almost surely equal to the radius of convergence of the non-random power series $\sum_{n=0}^{\infty} c_n z^n$.

**Problem 94.** (Ergodicity of product measure). This problem guides you to a proof of a different zero-one law.

(1) Consider the product measure space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \otimes_{\mathbb{Z}}\mu)$ where $\mu \in \mathcal{P}(\mathbb{R})$. Define $\tau : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ by $(\tau\omega)_n = \omega_{n+1}$. Let $\mathcal{I} = \{A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}}) : \tau(A) = A\}$. Then, show that $\mathcal{I}$ is a sigma-algebra (called the invariant sigma algebra) and that every event in $\mathcal{I}$ has probability equal to 0 or 1.

(2) Let $X_n, n \geq 1$ be i.i.d. random variables on a common probability space. Suppose $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a measurable function such that $f(x_1, x_2, \ldots) = f(x_2, x_3, \ldots)$ for any $(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$. Then deduce from the first part that the random variable $f(X_1, X_2, \ldots)$ is a constant, $a.s.$

[**Hint:** Approximate $A$ by cylinder sets. Use translation by $\tau^m$ to show that $\mathbf{P}(A) = \mathbf{P}(A)^2$.]

**Problem 95.** Consider the invariant sigma algebra and the tail sigma algebra. Show that neither is contained in the other.

**Problem 96.** (Chung-Erdös inequality).

(1) Let $A_i$ be events in a probability space. Show that

$$\mathbf{P}\left\{\bigcup_{k=1}^{n} A_k\right\} \geq \frac{(\sum_{k=1}^{n} \mathbf{P}(A_k))^2}{\sum_{k,\ell=1}^{n} \mathbf{P}(A_k \cap A_\ell)}$$

(2) Place $r_m$ balls in $m$ bins at random and count the number of empty bins $Z_m$. Fix $\delta > 0$. If $r_m > (1+\delta)m \log m$, show that $\mathbf{P}(Z_m > 0) \to 0$ while if $r_m < (1-\delta)m \log m$, show that $\mathbf{P}(Z_m > 0) \to 1$.

**Problem 97.** Give example of an infinite sequence of pairwise independent random variables for which Kolmogorov's zero-one law fails.

**Problem 98.** Let $X_i$, $i \in I$ be random variables on a probability space. Suppose that for some $p > 0$ and $M < \infty$ we have $\mathbf{E}[|X_i|^p] \leq M$ for all $i \in I$. Show that the family $\{X_i : i \in I\}$ is tight (by which we mean that $\{\mu_{X_i} : i \in I\}$ is tight, where $\mu_{X_i}$ is the distribution of $X_i$).

**Problem 99.** Let $X_i$ be i.i.d. random variables with zero mean and finite variance. Let $S_n = X_1 + \ldots + X_n$. Show that the collection $\{\frac{1}{\sqrt{n}} S_n : n \geq 1\}$ is tight. [**Note:** Tightness is essential for convergence in distribution. In the case at hand, convergence in distribution to $N(0,1)$ is what is called central limit theorem. We shall see it later.]

**Problem 100.** Suppose each of $r = \lambda n$ balls are put into $n$ boxes at random (more than one ball can go into a box). If $N_n$ denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left( \left| \frac{N_n}{n} - e^{-\lambda} \right| > \delta \right) \to 0$$

**Problem 101.** Let $X_n$ be i.i.d random variables such that $\mathbf{E}[|X_1|] < \infty$. Define the random power series $f(z) = \sum_{k=0}^{\infty} X_n z^n$. Show that almost surely, the radius of convergence of $f$ is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_n z^n$ is given by $(\limsup |c_n|^{\frac{1}{n}})^{-1}$].

**Problem 102.**
(1) Let $X$ be a real values random variable with finite variance. Show that $f(a) := \mathbf{E}[(X-a)^2]$ is minimized at $a = \mathbf{E}[X]$.

(2) What is the quantity that minimizes $g(a) = \mathbf{E}[|X-a|]$? [**Hint:** First consider $X$ that takes finitely many values with equal probability each].

**Problem 103.** Let $X_i$ be i.i.d. Cauchy random variables with density $\frac{1}{\pi(1+t^2)}$. Show that $\frac{1}{n} S_n$fils the weak law of large numbers by completing the following steps.

(1) Show that $t\mathbf{P}\{|X_1| > t\} \to c$ for some constant $c$.

(2) Show that if $\delta > 0$ is small enough, then $\mathbf{P}\{|\frac{1}{n-1} S_{n-1}| \geq \delta\} + \mathbf{P}\{|\frac{1}{n-1} S_{n-1}| \geq \delta\}$ does not go to 0 as $n \to \infty$ [*Hint:* Consider the possibility that $|X_n| > 2\delta n$].

(3) Conclude that $\frac{1}{n} S_n \xrightarrow{P} 0$. [*Extra:* With a little more effort, you can try showing that there does not exist deterministic numbers $a_n$ such that $\frac{1}{n} S_n - a_n \xrightarrow{P} 0$].

**Problem 104.** Let $X_n, X$ be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$, show that some subsequence $X_{n_k} \xrightarrow{a.s.} X$.

(2) If every subsequence of $X_n$ has a further subsequence that converges almost surely to $X$, show that $X_n \xrightarrow{P} X$.

**Problem 105.** For $\mathbb{R}^d$-valued random vectors $X_n, X$, the notions of convergence almost surely, in probability and in distribution are well-defined. If $X_n = (X_{n,1}, \ldots, X_{n,d})$ and $X = (X_1, \ldots, X_d)$, which of the following is true? Justify or give counterexamples.

(1) $X_n \xrightarrow{a.s.} X$ if and only if $X_{n,k} \xrightarrow{a.s.} X_k$ for $1 \leq k \leq d$.

(2) $X_n \xrightarrow{P} X$ if and only if $X_{n,k} \xrightarrow{P} X_k$ for $1 \leq k \leq d$.

(3) $X_n \xrightarrow{d} X$ if and only if $X_{n,k} \xrightarrow{d} X_k$ for $1 \leq k \leq d$.

**Problem 106.** Let $X_n, Y_n, X, Y$ be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (all r.v.s on the same probability space), show that $aX_n + bY_n \xrightarrow{P} aX + bY$ and $X_nY_n \xrightarrow{P} XY$. **[Hint:** You could try showing more generally that $f(X_n, Y_n) \to f(X, Y)$ for any continuous $f : \mathbb{R}^2 \to \mathbb{R}$.]

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{d} Y$ (all on the same probability space), then show that $X_nY_n \xrightarrow{d} XY$.

**Problem 107.** Let $X_n, Y_n, X, Y$ be random variables on a common probability space.

(1) Suppose that $X_n$ is independent of $Y_n$ for each $n$ (no assumptions about independence across $n$). If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and $U, V$ are independent. Further, $aX_n + bY_n \xrightarrow{d} aU + bV$.

(2) Give counterexample to show that the previous statement is false if the assumption of independence of $X_n$ and $Y_n$ is dropped.

**Problem 108.** If $X_n$ are independent random variables and $X_n$ converge in distribution. Show that the limit is degenerate.

**Problem 109.** If $X_n, Y_n$ are independent for each $n$ and $X_n + Y_n \xrightarrow{P} 0$. Show that there are numbers $y_n$ such that $X_n + y_n \xrightarrow{P} 0$.

**Problem 110.** For $\mathbb{R}^d$-valued random vectors $X_n, X$, we say that $X_n \xrightarrow{P} X$ if $\mathbf{P}(\|X_n - X\| > \delta) \to 0$ for any $\delta > 0$ (here you may take $\|\cdot\|$ to denote the usual norm, but any norm on $\mathbb{R}^d$ gives the same definition).

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $(X_n, Y_n) \xrightarrow{P} (X, Y)$.

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $X_n + Y_n \xrightarrow{P} X + Y$ and $\langle X_n, Y_n \rangle \xrightarrow{P} XY$. **[Hint:** Show more generally that $f(X_n, Y_n) \xrightarrow{P} f(X, Y)$ for any continuous function $f$ by using the previous problem for random vectors].

**Problem 111.** (1) If $X_n, Y_n$ are independent random variables on the same probability space and $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and $U, V$ are independent.

(2) If $X_n \xrightarrow{d} X$ and $Y_n - X_n \xrightarrow{P} 0$, then show that $Y_n \xrightarrow{d} X$.

**Problem 112.** Show that the sequence $\{X_n\}$ is tight if and only if $c_n X_n \xrightarrow{P} 0$ whenever $c_n \to 0$.

**Problem 113.** Suppose $X_n$ are i.i.d with $\mathbf{E}[|X_1|^4] < \infty$. Show that there is some constant $C$ (depending on the distribution of $X_1$) such that $\mathbf{P}\left(|n^{-1}S_n - \mathbf{E}[X_1]| > \delta\right) \leq Cn^{-2}$. (What is your guess if we assume $\mathbf{E}[|X_1|^6] < \infty$? You don't need to show this in the homework).

**Problem 114.** (1) (**Skorokhod's representation theorem**) If $X_n \xrightarrow{d} X$, then show that there is a probability space with random variables $Y_n, Y$ such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ and $Y_n \xrightarrow{a.s.} Y$. [**Hint:** Try to construct $Y_n, Y$ on the canonical probability space $([0,1], \mathcal{B}, \mu)$]

(2) If $X_n \xrightarrow{d} X$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that $f(X_n) \xrightarrow{d} f(X)$. [**Hint:** Use the first part]

**Problem 115.** Suppose $X_i$ are i.i.d with the Cauchy distribution (density $\pi^{-1}(1+x^2)^{-1}$ on $\mathbb{R}$). Note that $X_1$ is not integrable. Then, show that $\frac{S_n}{n}$ does not converge in probability to any constant. [**Hint:** Try to find the probability $\mathbf{P}(X_1 > t)$, and then use it].

**Problem 116.** Let $\{X_i\}_{i \in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) If $\{X_i\}_{i \in I}$ is uniformly integrable, then show that $\sup_i \mathbf{E}|X_i| < \infty$. Give a counterexample to the converse statement.

(2) Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function that goes to infinity and $\sup_i \mathbf{E}[|X_i| h(|X_i|)] < \infty$. Show that $\{X_i\}_{i \in I}$ is uniformly integrable. In particular, if $\sup_i \mathbf{E}[|X_i|^p] < \infty$ for some $p > 1$, then $\{X_i\}$ is uniformly integrable.

**Problem 117.** Let $X_n$ be i.i.d with $\mathbf{P}(X_1 = +1) = \mathbf{P}(X_1 = -1) = \frac{1}{2}$. Show that for any $\gamma > \frac{1}{2}$,

$$\frac{S_n}{n^\gamma} \xrightarrow{a.s.} 0.$$

[**Remark:** Try to imitate the proof of SLLN under fourth moment assumption. If you write the proof correctly, it should go for any random variable which has moments of all orders. You do not need to show this for the homework].

**Problem 118.** Let $X_n$ be independent real-valued random variables.

(1) Show by example that the event $\{\sum X_n$ converges to a number in [1,3]$\}$ can have probability strictly between $0$ and $1$.

(2) Show that the event $\{\sum X_n$ converges to a finite number$\}$ has probability zero or one.

**Problem 119.** Let $X_n$ be i.i.d exponential(1) random variables.

(1) If $b_n$ is a sequence of numbers that converge to $0$, show that $\limsup b_n X_n$ is a constant (a.s.). Find a sequence $b_n$ so that $\limsup\ b_n X_n = 1$ a.s.

(2) Let $M_n$ be the maximum of $X_1, \ldots, X_n$. If $a_n \to \infty$, show that $\limsup \frac{M_n}{a_n}$ is a constant (a.s.). Find $a_n$ so that $\limsup \frac{M_n}{a_n} = 1$ (a.s.).

[**Remark:** Can you do the same if $X_n$ are i.i.d N(0,1)? Need not show this for the homework, but note that the main ingredient is to find a simple expression for $\mathbf{P}(X_1 > t)$ asymptotically as $t \to \infty$].

**Problem 120.** Let $X_n$ be i.i.d real valued random variables with common distribution $\mu$. For each $n$, define the random probabilty measure $\mu_n$ as $\mu_n := \frac{1}{n}\sum_{k=1}^n \delta_{X_k}$. Let $F_n$ be the CDF of $\mu_n$. Show that

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \overset{a.s.}{\to} 0 \quad a.s.$$

**Problem 121.** Let $X_n$ be independent and $\mathbf{P}(X_n = n^a) = \frac{1}{2} = \mathbf{P}(X_n = -n^a)$ where $a > 0$ is fixed. For what values of $a$ does the series $\sum X_n$ converge a.s.? For which values of $a$ does the series converge absolutely, a.s.?

**Problem 122.** (**Random series**) Let $X_n$ be i.i.d $N(0,1)$ for $n \geq 1$.

(1) Show that the random series $\sum X_n \frac{\sin(n\pi t)}{n}$ converges a.s., for any $t \in \mathbb{R}$.

(2) Show that the random series $\sum X_n \frac{t^n}{\sqrt{n!}}$ converges for all $t \in \mathbb{R}$, a.s.

[**Note:** The location of the phrase "a.s" is all important here. Let $A_t$ and $B_t$ denote the event that the series converges for the fixed $t$ in the first or second parts of the question, respectively. Then, the first part is asking you to show that $\mathbf{P}(A_t) = 1$ for each $t \in \mathbb{R}$, while the second part is asking you to show that $\mathbf{P}(\cap_{t\in\mathbb{R}} B_t) = 1$. It is also true (and very important!) that $\mathbf{P}(\cap_{t\in\mathbb{R}} A_t) = 1$ but showing that is not easy.]

**Problem 123.** Suppose $X_n$ are i.i.d random variables with finite mean. Which of the following assumptions guarantee that $\sum X_n$ converges a.s.?

(1) (i) $\mathbf{E}[X_n] = 0$ for all $n$ and (ii) $\sum \mathbf{E}[X_n^2 \wedge 1] < \infty$.

(2) (i) $\mathbf{E}[X_n] = 0$ for all $n$ and (ii) $\sum \mathbf{E}[X_n^2 \wedge |X_n|] < \infty$.

**Problem 124.** (**Large deviation for Bernoullis**). Let $X_n$ be i.i.d Ber$(1/2)$. Fix $p > \frac{1}{2}$.

(1) Show that $\mathbf{P}(S_n > np) \leq e^{-np\lambda} \left(\frac{e^\lambda+1}{2}\right)^n$ for any $\lambda > 0$.

(2) Optimize over $\lambda$ to get $\mathbf{P}(S_n > np) \leq e^{-nI(p)}$ where $I(p) = -p\log p - (1-p)\log(1-p)$. (Observe that this is the *entropy* of the Ber($p$) measure introduced in the first class test).

(3) Recall that $S_n \sim \text{Binom}(n, 1/2)$, to write $\mathbf{P}(S_n = \lceil np \rceil)$ and use Stirling's approximation to show that
$$\mathbf{P}(S_n \geq np) \geq \frac{1}{\sqrt{2\pi np(1-p)}} e^{-nI(p)}.$$

(4) Deduce that $\mathbf{P}(S_n \geq np) \approx e^{-nI(p)}$ for $p > \frac{1}{2}$ and $\mathbf{P}(S_n < np) \approx e^{-nI(p)}$ for $p < \frac{1}{2}$ where the notation $a_n \approx b_n$ means $\frac{\log a_n}{\log b_n} \to 1$ as $n \to \infty$ (i.e., asymptotic equality on the logarithmic scale).

**Problem 125.** Carry out the same program for i.i.d exponential(1) random variables and deduce that $\mathbf{P}(S_n > nt) \approx e^{-nI(t)}$ for $t > 1$ and $\mathbf{P}(S_n < nt) \approx e^{-nI(t)}$ for $t < 1$ where $I(t) := t - 1 - \log t$.

**Problem 126.** Let $Y_1, \ldots, Y_n$ be independent random variables. A random variable $\tau$ taking values in $\{1, 2, \ldots, n\}$ is called a *stopping time* if the event $\{\tau \leq k\} \in \sigma(Y_1, \ldots, Y_k)$ for all $k$ (equivalently $\{\tau = k\} \in \sigma(Y_1, \ldots, Y_k)$ for all $k$).

(1) Which of the following are stopping times? $\tau_1 := \min\{k \leq n : S_k \in A\}$ (for some fixed $A \subseteq \mathbb{R}$). $\tau_2 := \max\{k \leq n : S_k \in A\}$. $\tau_3 := \min\{k \leq n : S_k = \max_{j \leq n} S_j\}$. In the first two cases set $\tau = n$ if the desired event does not occur.

(2) Assuming each $X_k$ has zero mean, show that $\mathbf{E}[S_\tau] = 0$ for any stopping time $\tau$. Assuming that each $X_k$ has zero mean and finite variance, show that $\mathbf{E}[S_1^2] \leq \mathbf{E}[S_\tau^2] \leq \mathbf{E}[S_n^2]$ for any stopping time $\tau$.

(3) Give examples of random $\tau$ that are not stopping times and for which the results in the second part of the question fail.

**Problem 127.** Let $X_k$ be independent random variables with zero mean and unit variance. Assume that $\mathbf{E}[|X_k|^{2+\delta}] \leq M$ for some $\delta < 0$ and $M < \infty$. Show that $S_n$ is asymptotically normal.

**Problem 128.** Let $X_k$ be i.i.d. random variables with zero mean and unit variance. Let $0 < a_1 < a_2 < \ldots$ be given numbers. Find sufficient conditions on $(a_i)_i$ such that $S_n$ is asymptotically normal.

**Problem 129.** Fix $\alpha > 0$.

(1) If $X, Y$ are i.i.d. random variables such that $\frac{X+Y}{2^{\frac{1}{\alpha}}} \stackrel{d}{=} X$, then show that $X$ must have characteristic function $\varphi_X(\lambda) = e^{-c|\lambda|^\alpha}$ for some constant $c$.

(2) Show that for $\alpha = 2$ we get $N(0, \sigma^2)$ and for $\alpha = 1$ we get symmetric Cauchy.

[*Note:* Only for $0 < \alpha \leq 2$ is $e^{-c|\lambda|^\alpha}$ a characteristic function. Hence a distribution with the desired property exists only for this range of $\alpha$].

**Problem 130.** Let $X_k$ be independent $\text{Ber}(p_k)$ random variables. If $\text{Var}(S_n)$ stays bounded, show that $S_n$ cannot be asymptotically normal.

**Problem 131** (*Weak law using characteristic functions*)**.** Let $X_k$ be i.i.d. random variables having characteristic function $\varphi$.

(1) If $\varphi'(0) = i\mu$, show that the characteristic function of $S_n/n$ converges to the characteristic function of $\delta_\mu$. Conclude that weak law holds for $S_n/n$.

(2) If $\frac{1}{n}S_n \stackrel{P}{\to} \mu$ for some $\mu$, then show that $\varphi$ is differentiable at 0 and $\varphi'(0) = i\mu$.

**Problem 132.** Find the characteristic functions of the distributions with the given densities.
(1) $e^{-|x|}$ for $x \in \mathbb{R}$, (2) $\frac{1}{2}\left(1 - \frac{|x|}{2}\right)_+$.

**Problem 133.** If $x_n \in \mathbb{R}$ and $e^{itx_n} \to 1$ for all $t \in \mathbb{R}$, then show that $x_n \to 0$.

**Problem 134.** Suppose $\mu_n, \mu$ are probability measures on $\mathbb{R}$ with characteristic functions $\varphi_n, \varphi$. If $\varphi_n(t) \to \varphi(t)$ for all $t \in \mathbb{Q}$, is it true that $\mu_n \to \mu$ weakly?

**Problem 135** (*Multidimensional central limit theorem*)**.** Let $X_n$ be i.i.d. $\mathbb{R}^d$-valued random vectors with zero mean and covariance matrix $\Sigma$. Let $S_n = X_1 + \ldots + X_n$. Show that $\frac{1}{\sqrt{n}}S_n \stackrel{d}{\to} N_d(0, \Sigma)$ using the replacement principle. Assume (for convenience) that third moments are finite (i.e., $\mathbf{E}[\|X_1\|^3] < \infty$).

**Problem 136.** Let $X_n$ be i.i.d. random variables with a non-degenerate distribution. If $S_n = X_1 + \ldots + X_n$, show that $\mathbf{P}\{|S_n| \leq M\} \to 0$ for any $M < \infty$.

**Problem 137.** For a real valued random variable $X$, its *concentration function* is defined as $Q_X(t) = \sup\{\mathbf{P}\{X \in [a, a+t] : a \in \mathbb{R}\}$, for $t \geq 0$ (so $Q_X(0)$ is the largest atom size in the distribution of $X$). If $X, Y$ are independent and $Z = X + Y$, show that $Q_{X+Y}(t) \leq Q_X(t)$ for all $t \geq 0$.

**Problem 138. [3 marks each]** For each of the following statements, state whether they are true or false, and justify or give counterexample accordingly.

(1) If $\mu, \nu$ are Borel probability measures on $\mathbb{R}$ and $\mu \ll \nu$, then either $\nu \perp \mu$ or $\nu \ll \mu$.

(2) If $\sum_n X_n$ converges a.s. and $\mathbf{P}(Y_n = X_n) = 1 - \frac{1}{n^2}$. Then $\sum_n Y_n$ converges a.s.

(3) If $\{X_n\}$ is an $L^2$ bounded sequence of random variables, and $\mathbf{E}[X_n] = 1$ for all $n$, then $X_n$ cannot converge to zero in probability.

(4) If $X_n \xrightarrow{d} X$, then $X_n^2 \xrightarrow{d} X^2$.

(5) Suppose $X_n$ are independent with $\mathbf{E}[X_n] = 0$ and $\sum \text{Var}(X_n) = \infty$. Then, almost surely $\sum X_n$ does not converge.

(6) Suppose $X_n, Y_n$ are random variables such that $|X_n| \leq |Y_n|$ for all $n$. If $\sum Y_n$ converges almost surely, then $\sum X_n$ converges almost surely.

**Problem 139. [2 marks+4 marks + 4 marks]** Let $X, Y$ be random variables on a common probability space. Assume that both $X$ and $Y$ have finite variance.

(1) Show that $\mathbf{E}[(X-a)^2]$ is minimized uniquely at $a = \mathbf{E}[X]$.

(2) Find values of $a, b$ that minimize $f(a,b) = \mathbf{E}\left[(Y - a - bX)^2\right]$. Are they unique?

(3) Suppose $\mathbf{P}(X = k) = \frac{1}{10}$ for $k = 1, 2 \ldots, 10$. At what value(s) of $a$ is $\mathbf{E}[|X-a|]$ minimized? Is the minimizer unique?

**Problem 140. [10 marks]** Let $G_1, G_2, \ldots$ be i.i.d Geometric($p$) random variables (this means $\mathbf{P}(G_1 = k) = p(1-p)^{k-1}$ for $k \geq 1$). Let $X_1, X_2, \ldots$ be i.i.d random variables with $\mathbf{E}[|X_1|] < \infty$. Define $N_k := G_1 + G_2 + \ldots + G_k$. Show that as $k \to \infty$,

$$\frac{X_1 + X_2 + \ldots + X_{N_k}}{k} \xrightarrow{P} \frac{1}{p}\mathbf{E}[X_1]$$

**Problem 141. [5 marks+5 marks]** Let $U_k, V_k$ be i.i.d Uniform([0,1]) random variable.

(1) Show that $\sum_k U_k^{\frac{1}{k}} - V_k^{\frac{1}{k}}$ converges a.s.

(2) Let $S_n = U_1 + U_2^2 + \ldots + U_n^n$. Show that $S_n$ satisfies a CLT. In other words, find $a_n, b_n$ such that $\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1)$.

**Problem 142. [5 marks+5 marks]** Let $\mathbf{Z}^{(n)} = (Z_1^{(n)}, \ldots, Z_n^{(n)})$ be a point sampled uniformly from the sphere $S^{n-1}$ (this means that $\mathbf{P}(\mathbf{Z}^{(n)} \in A) = \text{area}(A)/\text{area}(S^{n-1})$ for any Borel set $A \subseteq S^{n-1}$).

(1) Find the density of $Z_1^{(n)}$.

(2) Using (1) or otherwise, show that $\sqrt{n}Z_1^{(n)} \xrightarrow{d} N(0,1)$ as $n \to \infty$.

[**Hint:** One way to generate $\mathbf{Z}^{(n)}$ is to sample $X_1, \ldots, X_n$ i.i.d N(0,1) and to set $\mathbf{Z}^{(n)} = \frac{1}{\|X\|}(X_1, \ldots, X_n)$ where $\|X\| = \sqrt{X_1^2 + X_2^2 + \ldots + X_n^2}$. You may assume this fact without having to justify it].

**Problem 143. [5 marks+5 marks]**

(1) Let $\mu$ be a probability measure on $\mathbb{R}$ with characteristic function $\hat{\mu}(t)$. Then, show that for any $t_1, t_2, \ldots, t_n \in \mathbb{R}$, the $n \times n$ matrix $A$ with entries $a_{i,j} = \hat{\mu}(t_i - t_j)$ is non-negative definite.

(2) Suppose $|\hat{\mu}(t_0)| = 1$ for some $t_0 \neq 0$. Then, $\mu$ is supported on a lattice, that is, $\mu(a\mathbb{Z} + b) = 1$ for some $a, b \in \mathbb{R}$. [**Hint:** Use part (1) with $n = 2$ and appropriate $t_1, t_2$].

**Problem 144. [10 marks]** Let $X_1, X_2, \ldots$ be i.i.d Bernoulli$\left(\frac{1}{2}\right)$ random variables. For each $n \geq 1$, define $L_n$ to be the longest run of ones in $(X_1, \ldots, X_n)$, that is,

$$L_n := \max\{k : \exists j \leq n - k \text{ such that } X_{j+1} = X_{j+2} = \ldots = X_{j+k} = 1\}.$$

Prove that $\frac{L_n}{\log n} \xrightarrow{P} \frac{1}{\log 2}$.

## NEW PROBLEMS ADDED IN 2019

**Problem 1.** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

(1) Let $S \subseteq \mathcal{F}$ be a finite collection of sets. Then show that $\sigma(S)$ is also a finite collection.

(2) Let $\mathcal{A}$ be an algebra that generates $\mathcal{F}$. For any $A \in \mathcal{F}$ and any $\epsilon > 0$, show that there is some $B \in \mathcal{A}$ such that $\mu(A \Delta B) < \epsilon$ (here $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference).

**Problem 2.** (1) Let $S$ be the set of all $x \in [0,1]$ whose base $b$-expansion contains all the digits $0, 1, \ldots, b-1$, for every $b \in \{2, 3, 4 \ldots\}$. Show that $\lambda(S) = 1$, where $\lambda$ is the Lebesgue measure on $[0,1]$.

(2) Let $S$ be the set of all points in $\mathbb{R}^2$ that can be written as a convex combination of two rational points (a rational point is one whose co-ordinates are all rational numbers). Show that $S$ has zero Lebesgue measure.

**Problem 3.** Let $A, B$ be Borel subsets of $[0,1]$ and let $C \subseteq \mathbb{R}^2$ be the union of all line segments connecting a point in $A \times \{0\}$ to a point in $\{0\} \times B$. Is $C$ measurable? If $\lambda(A) = \lambda(B) = 0$, it it necessarily true that $\lambda_2(C) = 0$? If $\lambda(A) > 0$ and $\lambda(B) > 0$, is it necessarily true that $\lambda_2(C) > 0$?

**Problem 4.** Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $\mathbf{E}[X^p] < \infty$ and $\mathbf{E}[X^{-p}] < \infty$ for some $p > 0$. Then show that $\log X$ is integrable and that

$$\frac{1}{q}\mathbf{E}[X^q - 1] \to \mathbf{E}[\log X] \quad \text{as } q \downarrow 0.$$

**Problem 5.** A random interval is one whose end points are picked independently and uniformly from $[0,1]$. If $I_1, \ldots I_n$ are $n$ i.i.d. random intervals, what is the probability that they have a point of intersection?

**Problem 6.** A random interval is one whose end points are picked independently and uniformly from $[0,1]$. If $I_1, \ldots I_n$ are $n$ i.i.d. random intervals, what is the probability that one of them intersects all the others?