Problem set 1
Due date: 16th Aug

Submit the starred exercises only

Exercise 1 (*). For each of the “random experiments” described below, describe the sample space and the probabilities. Also compute the probability of the event \( A \) specified. If no event is specified, just give the probability space.

1. A fair die is thrown until a 6 or a 1 shows up. Let \( A \) be the event that the number of throws is at least \( n \).
2. A coin is tossed and a die is thrown. \( A \) is the event that either the coin turns up head or the die shows up an even number.
3. Place \( k \) unlabeled balls in \( n \) labelled urns. Let \( A \) be the event that the first urn is empty. [Note: See the next part before you think you have solved this.]
4. Place \( k \) unlabeled balls in \( n \) labelled urns so that all distinguishable configurations are equally likely. Let \( A \) be the event that the first urn is empty.

Exercise 2 (*). Let \( A_1, \ldots, A_n \) be events in a probability space \((\Omega, p)\) and let \( m \leq n \). Let \( B_m \) be the event that at least \( m \) of the events \( A_1, \ldots, A_n \) occur. That is

\[
B_m = \bigcup_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} (A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}).
\]

In class we showed by the inclusion exclusion formula that

\[
P(B_1) = S_1 - S_2 + \ldots + (-1)^{n-1} S_n
\]

where

\[
S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).
\]

Show that

\[
P(B_m) = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \ldots + (-1)^{n-m} \binom{n-1}{m-1} S_n.
\]

[Note: Also see Problem 4 and the remark at the end of it.]

Exercise 3. For each of the “random experiments” described below, describe the sample space and the probabilities. Also compute the probability of the event \( A \) specified. If no event is specified, just give the probability space.

1. A die is thrown until the first time a head is immediately followed by a tail (e.g., if the tosses are \( TTHHT \) then we needed 5 tosses). Find the probability that at least \( n \) throws are needed.
2. Place \( k \) labeled balls in \( n \) unlabeled urns. Let \( A \) be the event that the first ball and the second ball are in distinct urns. Do it for both cases - (a) Each distinguishable distribution is equally likely and (b) All distributions are equally likely (even if not distinguishable).
3. Place \( k \) unlabeled balls in \( n \) unlabeled urns.
(4) 13 cards are dealt from a shuffled deck off 52 cards. Let \( A \) be the event that the cards dealt contains a series.\(^1\) Let \( B \) be the event that the cards dealt contains a set.\(^2\)

(5) A drunkard returns home with a bunch of \( n \) keys in his pocket. He randomly tries them one after another till the lock opens. Let \( A \) be the event that the fifth key opens the lock. Assume that he does not try the same key twice.

**Exercise 4.** Let \( A_1, \ldots, A_n \) be events in a probability space \( (\Omega, \mathcal{P}) \) and let \( m \leq n \). Let \( C_m \) be the event that exactly \( m \) out of the \( n \) events \( A_1, \ldots, A_n \) occur. That is

\[
C_m = \bigcup_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} \left\{ \left( \bigcap_{j=1}^{m} A_{i_j} \right) \bigcap \left( \bigcap_{k \notin \{i_1, \ldots, i_m\}} A_k^c \right) \right\}.
\]

\[
\mathbf{P}(C_m) = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \ldots + (-1)^{n-m} \binom{n}{m} S_n.
\]

[Note: If you solve one of Problem 2 or Problem 4, you can solve the other using the relationship \( C_m = B_m \setminus B_{m+1} \) or \( B_m = C_m \sqcup C_{m+1} \sqcup \ldots \sqcup C_n \).]

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\(^1\)A series means three cards of the same suit in succession, eg., 9,10,J of spades. Here ace is interpreted as 1 and hence A,2,3 is a series but not QKA or KA2.

\(^2\)A set means three cards of distinct suits but having the same value, eg., the queen of spades, diamonds and hearts or the 7 of spades, hearts and clubs.