Problem set 3

Due date: **16th Sep**

Submit the starred exercises only

Exercise 10 (*). When the CDF is not very explicit, it is useful to have simple bounds for probabilities. Here are some inequalities for Normal probabilities.

(1) If $X \sim N(0,1)$, show that $\mathbf{P}\{X \in [a,b]\} \le \frac{b-a}{\sqrt{2\pi}}$ (it is only useful when the right side is less than 1).
(2) If $X \sim N(0,1)$, show that $\mathbf{P}\{X > t\} \le \frac{1}{t\sqrt{2\pi}} e^{-t^2/2}$ for any $t > 0$. [**Hint:** For $u > t$, we have $e^{-u^2/2} \le \frac{u}{t} e^{-u^2/2}$].
(3) Compute the bound for $t = 1,2,3,4$ and compare with the actual probability (which can be found from Normal probability tables).

Exercise 11 (*). Let $U \sim \text{unif}([0,1])$ and let F be a CDF. In class we saw that if F is strictly increasing and continuous, then $X := F^{-1}(U)$ is a random variable whose CDF is F. When F is not strictly increasing or is not continuous, the inverse function does not exist. Here is how to modify the idea. Define,

$$G(t) := \inf\{x : F(x) \ge t\}, \quad \text{for } t \in [0,1]$$

(1) Then show that $-\infty < G(t) < +\infty$ for all $t \in (0,1)$. (Observe that $G(0) = -\infty$ always, while $G(1)$ may be finite or equal to $+\infty$).
(2) Show that G is non-decreasing and left continuous.
(3) Let $X = G(U)$. Then show that $X \sim F$. [**Hint:** Argue that $G(t) \le x$ if and only if $F(x) \ge t$.].

[**Remark:** To really understand the definition of G, draw the graphs of a few CDFs that have jumps and intervals of constancy. When F is strictly increasing and continuous, G is just the inverse of F].

Exercise 12. Let $\nu \ge 1$ be an integer and let $\lambda > 0$. Show that the Gamma(ν,λ) distribution has CDF

$$F(t) = 1 - e^{-\lambda t} \sum_{k=0}^{\nu-1} \frac{\lambda^k t^k}{k!}.$$

[**Remark:** Observe that the right hand side here is exactly $\mathbf{P}\{N \ge \nu\}$ where $N \sim \text{Pois}(\lambda)$. This is an important connection between the Gamma and Poisson distributions that will reappear later in the course.].

Exercise 13. Let X be a random variable with p.m.f f or with p.d.f f. Then the point(s) where f attains its maximum value is called the mode(s).

(1) Let $X \sim \text{Bin}(n,p)$. Show that there are either one or two modes and find them.
(2) Let $X \sim \text{Pois}(\lambda)$. Show that there is only one mode and find it.
(3) Let $X \sim \text{Gamma}(\nu,\lambda)$. For $\nu \ge 1$, show that there is only one mode and find an equation for it. What happens for $0 < \nu < 1$.

Exercise 14. In this exercise you will be guided to a proof of the identity

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ \text{ for } \alpha,\beta>0.$$

(1) Observe that $\left(\int_a^b f(t)dt\right)\left(\int_c^d g(s)ds\right)=\int_a^b\int_c^d f(t)g(s)dsdt$. Use this to write $\Gamma(\alpha)\Gamma(\beta)$ as a double integral.

(2) Make a change of variables $x=t+s$, $y=t/(t+s)$ and apply the change of variable formula (to be done in class soon). Deduce the desired identity.

Exercise 15. Here are some general properties of CDFs that we did not cover in class.

(1) Show that any increasing function $F:\mathbb{R}\to\mathbb{R}$ can have atmost a countable number of discontinuities. In particular, a CDF can have only a countable number of discontinuities. [**Hint:** Each jump must jump over a rational number.]

(2) An increasing function $F:\mathbb{R}\to\mathbb{R}$ is called *atomic* if there is a countable set S and non-negative numbers w_x, $x\in S$, such that $F(t)=\sum_{x\in S\cap(-\infty,t]} w_x$. Show that any CDF F can be written as $F=F_1+F_2$ where F_1,F_2 are increasing and F_1 is atomic and F_2 is continuous.