Problem set 6

Due date: **24th Nov**

Submit the starred exercises only

Exercise 32 (*). Let X be a non-negative random variable.

(1) **Second moment method:** Show that $\mathbf{P}(X > 0) \geq \frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]}$.
 [**Hint:** Write $X = X \cdot \mathbf{1}_{X>0}$].

(2) Let X_n be a sequence of random variables such that $\mathbf{E}[X_n] = n$ and $\mathbf{E}[X_n^2] \leq Cn^2$. Use the second moment method to show that $\mathbf{P}(X_n > 0) \geq \frac{1}{C}$, a uniform lower bound on the probability that X_n is strictly positive.

Remark: In contrast, Chebyshev's inequality can be used to get $\mathbf{P}(X_n = 0) \leq \mathbf{P}(|X_n - \mathbf{E}[X_n]| \geq n) \leq C - 1$ and hence $\mathbf{P}(X_n > 0) \geq 2 - C$ which is fine when $C < 2$ but useless when $C \geq 2$ as probabilities are always non-negative!

Exercise 33 (*). X is a non-negative r.v such that $\mathbf{E}[X^n] \leq 2^n$ for all $n \geq 1$. Show that $\mathbf{P}(X \leq 2) = 1$. (The converse is obvious).

Exercise 34. If $X \sim \text{Pois}(\lambda)$ and $Y \Big|_X \sim \text{Bin}(X, p)$, then show that $Y \sim \text{Pois}(\lambda p)$.

Exercise 35 (*). (**Branching processes**) Informally, we have one individual (the zeroeth generation). It dies after giving birth to a random number of offsprings (who constitute the first generation). Then, each individual in the first generation gives birth to a random number of offsprings and dies. All these grandchildren constitute the second generation. The process continues this way. The number of offsprings of different individuals (in the same or different generation) are independent and identically distributed. The question is to determine whether the population necessarily becomes extinct (note that if the number of individuals in some generation becomes zero, the process stops).

Mathematically, let $L, L_{n,i}$, $i \geq 1$, $n \geq 0$ be i.i.d random variables taking values in non-negative integers. Let $Z_0 = 1$. Inductively for $n \geq 1$ define

$$Z_n = \begin{cases} L_{n,1} + \ldots + L_{n,Z_{n-1}} & \text{if } Z_{n-1} \geq 1. \\ 0 & \text{if } Z_{n-1} = 0. \end{cases}$$

Let $\mathbf{E}[L] = m$ and $\text{Var}(L) = \sigma^2 < \infty$.

(1) Show that $\mathbf{E}[Z_n] = m^n$.

(2) Show that $\text{Var}(Z_n) = \sigma^2(m^{n-1} + m^n + \ldots + m^{2n-3} + m^{2n-2})$.

(3) If $m < 1$, use Markov's inequality to show that $\mathbf{P}(Z_n > 0) \to 0$. Since $\{Z_n > 0\}$ is a decreasing sequence of events, conclude that $\mathbf{P}(\text{eventual extinction of population}) = 1$.

(4) If $m > 1$, use the second moment method to show that there is some $\delta > 0$ such that $\mathbf{P}(Z_n \geq 1) \geq \delta$. Conclude that $\mathbf{P}(\text{eventual extinction of the population}) < 1$ or equivalently, $\mathbf{P}(\text{survival forever of the population}) > 0$.

(5) (Do not submit) If $m > 1$ but $\sigma^2 = \infty$, deduce that we continue to have $\mathbf{P}(\text{survival forever of the population}) > 0$.

Remark: The case when $m = 1$ is more delicate, but it can be shown that $\mathbf{P}(\text{eventual extinction of population}) = 1$ in that case also.