Schramm Löwner Evolutions

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ABSTRACT. Lecture notes from SLE seminar in Summer and Fall of 2010 at IISc, Bangalore. Mainly, we follow (some chapters of) Wendelin Werner's St. Flour lecture notes on SLE, but filling in most proofs. Much of this latter is in Greg Lawler's book from which we borrow a lot of things. Some proofs are a little different. However, we want to keep it streamlined to a minimal introduction to SLE. In particular, we stick to chordal versions of all theorems.

Contents

CHAPTER 1

Complex analysis background

1.1. Boundary behaviour of conformal maps

When does a conformal map on a region extends continuously to the boundary point? We present two results - Schwarz's reflection principle which applies to general holomorphic function, and Theorem 1.1.3 which applies only to conformal maps.

Theorem 1.1.1 (Schwarz's reflection principle)**.** *Let* $f : \mathbb{D}_+ \to \mathbb{H}$ *be analytic and suppose that* $\operatorname{Im} f(z) \to 0$ *as* $\operatorname{Im} z \to 0$*. Then,* f *extends as an analytic function to* $\mathbb{D}$ *and the extension satisfies* $\overline{f(z)} = f(\bar{z})$*.*

Needless to say, one can also perform reflection along other curves, for example if $I \subset \partial\Omega$ is a circular arc and f is an analytic function on Ω such that $\operatorname{Im} f(z) \to 0$ (or if $|f(z)| \to 1$) as $z \to I$, then f extends as an analytic function across I by reflection. This can be deduced from Theorem 1.1.3 by mapping I to a line segment by a conformal map. For example, if $I \subset \mathbb{T}$, then the map $w \to i\frac{1+w}{1-w}$ works.

Definition 1.1.2. Let Ω be a region. We say that $z_n \in \Omega$ converge to $\zeta \in \partial\Omega$ *along a curve* if there exists a curve $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(t) \in \Omega$ if $t < 1$ and $\gamma(1) = \zeta$ and $\gamma(t_n) = z_n$ for some $t_1 < t_2 < \ldots$.

Declare $(z_n) \sim (w_n)$ if the interlaced sequence $z_1, w_1, z_2, w_2, \ldots$ converges to a boundary point ζ along a curve. We call the equivalence classes as *prime ends*. (The usual definition is more generally applicable, but we need only this much. This is because boundary points that we consider will have at least one prime end corresponding to them).

Theorem 1.1.3 (Rudin 14.18)**.** *Let* Ω *be a bounded s.c. region. Let* $f : \Omega \to \mathbb{D}$ *be any Riemann map (one-one onto conformal map).*

(1) *If* $z_n \to \zeta$ *along a curve, then* $f(z_n)$ *has a limit that belongs to* $\partial\mathbb{D} = \mathbb{T}$*. The limit is the same for all sequences in the same prime end.*
(2) *The limits thus obtained are different for distinct prime ends.*

Remark 1.1.4. The proof of Theorem 1.1.3 given in Rudin's book depends on a result that *'a bounded holomorphic function* f *on* $\mathbb{D}$ *has radial limits at a.e point of the circle'*. This is a result of interest in itself and is proved using maximal functions (see chapter 11 of Rudin). For the purpose of proving Theorem 1.1.3, it clearly suffices to show that *'for a.e. point on the circle, there is some curve converging to it from inside the disk, along which the limit of f exists'*. This fact will fall out of other probabilistic considerations later.

Riemann mapping theorem asserts that given a s.c. $\Omega \neq \mathbb{C}$, there does exist a one-one onto conformal map $f : \Omega \to \mathbb{D}$. The space of all such conformal maps is three-dimensional, since the group of conformal automorphisms of the disk is. For example, to get uniqueness, specify interior points $z_0 \in \Omega$ and $0 \in \mathbb{D}$ and ask $f(z_0) = 0$

and $f'(z_0) > 0$. Any three constraints should do as well, and as a consequence of Theorem 1.1.3, these constraints can be on the boundary.

Corollary 1.1.5. *Let Ω be a bounded s.c. region. Let $f:\Omega\to\mathbb{H}$ be any Riemann map (one-one onto conformal map). Then the conclusions of Theorem 1.1.3 are valid, with $\partial\mathbb{H}=\mathbb{R}\bigcup\{\infty\}$ in place of $\mathbb{T}$.*

PROOF. $\psi(z) := \frac{z-i}{z+i}$ maps $\mathbb{H}$ onto $\mathbb{D}$ conformally and is a continuous bijection from $\overline{\mathbb{H}}$ onto $\overline{\mathbb{D}}$. Apply Theorem 1.1.3 to $\psi\circ f$ and tranfer results back by ψ^{-1}. ■

1.2. The standard setting

Definition 1.2.1. A bounded set $K\subset\mathbb{H}$ is said to be a *hull* if $K=\overline{K}\cap\mathbb{H}$ and $\mathbb{H}\backslash K$ is s.c. Let $\mathrm{r}(K)=\sup\{|z|:z\in K\}$.

Example 1.2.2. $\mathbb{D}_+$, $(0,i]$ and $(1,1+i]\bigcup(-1,1+2i]$ etc, are hulls while $\{1+iy\ :\ y>0\}$ or $2i+\mathbb{D}$ or $\{z:x\ge 1,\ xy\le 1\}$ etc., are not.

Theorem 1.2.3 (Riemann mapping theorem)**.** *Let K be a hull. Then, there exists a unique conformal map $\Phi:\mathbb{H}\backslash K\to\mathbb{H}$ such that $\Phi(z)=z+\frac{a}{z}+O(z^{-2})$ as $z\to\infty$. The number a is non-negative.*

PROOF. By Riemann mapping theorem, there is a conformal map $\Phi:\mathbb{H}\backslash K\to\mathbb{H}$. Corollary 1.1.5 asserts that $t:=\Phi(\infty)$ is a well-defined number in $\mathbb{R}\bigcup\{\infty\}$. Post-compose by $z\to\frac{z}{t^{-1}z-1}$ which maps $\mathbb{H}$ conformally onto itself to get a conformal $\Psi:\mathbb{H}\backslash K\to\mathbb{H}$ such that $\Psi(\infty)=\infty$.

Since K is bounded, for all real x with $|x|>x_0=\mathrm{r}(K)$, as $z\to x$ from above, we get $f(z)\to\mathbb{R}$ from above. Schwarz's reflection principle applies and Ψ extends analytically to $\mathbb{C}\backslash(K\bigcup\overline{K})$ and the extension maps $(-\infty,-x_0]\bigcup[x_0,\infty)$ into $\mathbb{R}$. We want to take the power series expansion of Ψ at ∞. More precisely, set $F(w)=1/\Psi(1/w)$ for $|w|<1/\mathrm{r}(K)$ and write $F(w)=b_1w+b_2w^2+\dots$ as $F(0)=0$. Since F maps reals to reals, b_i are all real numbers.

Thus, for $|z|>\mathrm{r}(K)$, we can write $\Psi(z)=F(1/z)^{-1}=a_{-1}z+a_0+a_1z^{-1}+a_2z^{-2}+\dots$ where a_i are real, which is what we mean by the power series of Ψ at ∞. Moreover $a_{-1}>0$ or else Ψ would map parts of upper half plane into the lower half plane. As $a_{-1}>0$ and a_0 is real, $w\to(w-a_0)/a_{-1}$ maps $\mathbb{H}$ conformally onto itself. Compose with Ψ to get the conformal map $\Phi(z)=z+\frac{a}{z}+O(z^{-2})$ where $a=a_1/a_{-1}$. The uniqueness is also clear.

It remains to show that $a\ge 0$. This is really a boundary version of Schwarz's lemma. For $z_0\in\mathbb{H}$, let $\psi_{z_0}(z):=(z-z_0)/(z-\overline{z}_0)$ is a Riemann map from $\mathbb{H}$ onto the unit disk taking z_0 to 0. Let $w_0:=\Psi(z_0)$. Then if $g:=\psi_{w_0}\circ\Phi\circ\psi_{z_0}^{-1}$, then g^{-1} maps $\mathbb{D}$ into $\mathbb{D}$ with $g(0)=0$. Hence by Schwarz's lemma, $|(g^{-1})'(0)|\le 1$ with equality if and only if $g(z)=z$. Thus,

$$1\le|g'(0)|=|\psi'_{w_0}(w_0)||\Phi'(z_0)||\psi'_{z_0}(z_0)|=\frac{(\mathrm{Im}\,z_0)\,|\Phi'(z_0)|}{\mathrm{Im}\,\Phi(z_0)}=\frac{1+ay_0^{-2}+O(y_0^{-3})}{1-ay_0^{-2}+O(y_0^{-3})}$$

where in the end we chose $z_0=iy_0$. Letting $y_0\to\infty$, we see that a must be non-negative. Further, $a=0$ if and only if K is empty. ■

Remark 1.2.4. Schwarz's lemma (or Schwarz-Pick lemma) has the following meaning. Consider the disk $\mathbb{D}$ with the hyperbolic metric $(1-|z|^2)^{-2}(dx^2+dy^2)$. If $f:\mathbb{D}\to\mathbb{D}$ is analytic, the push-forward of the metric is $\frac{|f'(z)|^2}{(1-|f(z)|^2)^2}(dx^2+dy^2)$. Schwarz-Pick lemma is the statement that any analytic function $f : \mathbb{D} \to \mathbb{D}$ is a contraction in the hyperbolic metric, and if it is not a strict contraction at some z, then f is a Möbius transformation of the disk onto itself, which is an isometry of the hyperbolic metric.

It is easy to check that $\mathbb{H}$ with the metric $\frac{dx^2+dy^2}{y^2}$ is isomorphic to the the disk with its hyperbolic metric, and hence Schwarz-Pick lemma gives that for any holomorphic $g : \mathbb{H} \to \mathbb{H}$, we have $\frac{|g'(z)|}{\operatorname{Im} g(z)} \le \frac{1}{\operatorname{Im} z}$ which is what we used above.

Example 1.2.5. Suppose $K = [x, x+it]$ where x is real and $t > 0$. Then $\Phi_K(z) = \sqrt{(z-x)^2+t^2}$ where the principal brach of square root $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ for $\theta \in (0,2\pi)$. We can write the expansion $\Phi_K(z) = z + \frac{t^2/2}{z} + O(z^{-2})$. For $K = r\overline{\mathbb{D}}_+ := \{z \in \mathbb{H} : |z| \le r\}$ we can see that $\Phi_K(z) = z + \frac{r^2}{z}$. A not so trivial exercise is to find Φ_K for the oblique slit $[0, te^{i\theta}]$ for some $0 < \theta < \pi$ and $t > 0$.

1.3. Brownian motion

What is Brownian motion? We give a quick definition and a convenient way to visualize it. It can be shown that Brownian motion does exist.

Definition 1.3.1. Brownian motion in the plane is $W = (W_t)_{t\ge 0}$ is a collection of complex-valued random variables on a probability space such that (a) $W_0 = 0$ w.p.1. (b) For any $t_1 < t_2 < \ldots < t_k$ the random variables $W_{t_i} - W_{t_{i-1}}$ are independent, and for any $s < t$ the real and imaginary parts of $W_t - W_s$ are independent $N(0, t-s)$ distributed. (c) The function $t \to W_t$ is a continuous function w.p.1.

A convenient way to visualize Brownian motion is to fix $h > 0$, imagine a particle which picks a random direction and moves a distance $\sqrt{2h}$ in a time duration of h, then picks a random direction again and moves a distance $\sqrt{2h}$ etc. As $h \to 0$, this random trajectory converges to Brownian motion in a precise sense.

What we really use in the sequel are the following basic properties of Brownian motion.

(1) *Symmetries:* $e^{i\theta}W(\cdot) \overset{d}{=} W(\cdot)$ (rotation invariance). $W(r\cdot) \overset{d}{=} \sqrt{r}W(\cdot)$ (scale invariance). $W \overset{d}{=} -W$ (reflection invariance).
(2) *Strong Markov property:* Let τ be a stopping time. Then, conditional on $(W_{t\wedge\tau})_{t\ge 0}$, the future path $W(\tau+\cdot) - W(\tau)$ is a standard Brownian motion.

Our interest in Brownian motion for now is because of its usefulness as a tool to prove things about harmonic functions and related things! Here is the basic result that connects Brownian motion to harmonic functions and the Dirichlet problem.

Theorem 1.3.2. *Let $\Omega \subseteq \mathbb{C}$ be a region such that Ω^c has a connected component containing more than one point*[1]. *Let $\tau := \inf\{t : W_t \notin \Omega\}$. Let $f : \partial\Omega \to \mathbb{R}$ be a bounded measurable function.*

(1) $P_z(\tau < \infty) = 1$ *for any $z \in \Omega$ and $u(z) := \mathbf{E}[f(W_\tau)]$ is well defined and harmonic in Ω.*
(2) *Assume that f is continuous at a point $\zeta \in \partial\Omega$ and that there is a connected set $K_\zeta \subseteq \Omega^c$ that contains ζ and has more than one point. Then $u(z) \to f(\zeta)$ as $z \to \zeta$.*

[1]Here and elsewhere, we think of Ω as a subset of the sphere $\mathbb{C}\bigcup\{\infty\}$, so ∞ is included in Ω^c.

(3) *Assume that for every* $\zeta \in \partial\Omega$, *there is a set* K_ζ *as in (2). Then, for any* $f \in C(\partial\Omega)$, *the function* u *is the unique solution to the*

Dirichlet problem: *Find* $v \in C(\overline{\Omega})$ *such that* $\Delta v = 0$ *in* Ω, *and* $v = f$ *on* $\partial\Omega$.

PROOF. (1) We leave the part about $P_z(\tau < \infty) = 1$ to Exercise 1.3.3. Fix z and r such that $z + r\overline{\mathbb{D}} \subseteq \Omega$. Define the stopping time $T := \inf\{t : |W_t - z| = r\}$. By strong Markov property of W we get $\mathbf{E}\left[u(z) \,\middle|\, W_{\cdot\wedge T}\right] = u(W_T)$. Take expectations again to get $u(z) = \mathbf{E}[u(W_T)] = \int_0^{2\pi} u(z + re^{i\theta})\frac{d\theta}{2\pi}$ by rotation invariance of W. Thus u has mean-value property. Therefore $\Delta u = 0$ in Ω.

(2) Fix $\epsilon > 0$ and pick $\delta < \text{dia}(K_\zeta)$ so that $|f(\zeta) - f(\xi)| < \epsilon$ for any $\xi \in \partial\Omega \cap B(\zeta,\delta)$. If $z \to \zeta$, then $\mathbf{P}_z\left\{W \text{ exits } B(\zeta,\delta) \text{ before hitting } K_\zeta\right\} \to 0$ [*Why?*]. On the complement of this event $W_\tau \in B(\zeta,\delta)$ and hence $|f(W_\tau) - f(\zeta)| < \epsilon$. Taking expectations, we get

$$\mathbf{E}[|f(W_\tau) - f(\zeta)|] \le \epsilon + 2\|f\|\mathbf{P}_z\left\{W \text{ exits } B(\zeta,\delta) \text{ before hitting } K_\zeta\right\} \to 0$$

as $z \to \zeta$. That is $u(z) \to f(\zeta)$, hence $u \in C(\overline{\Omega})$ and $u = f$ on $\partial\Omega$.

(3) u is a solution by part (2). Uniqueness is by maximum principle. ■

Exercise 1.3.3. Let $\Omega \subseteq \mathbb{C}$ be a region such that Ω^c has a connected component containing more than one point and let $\tau := \inf\{t : W_t \notin \Omega\}$. Then, $\mathbf{P}_z(\tau < \infty) = 1$ for any $z \in \Omega$. [**Hint:** By symmetry, there is a positive probability that W started at a point inside $\mathbb{D}(z,r)$ exits the disk through a given arc $I \subset \partial\mathbb{D}(z,r)$. Try to construct a sequence of disks and arcs so that if W exits those disks in those arcs, then it hits Ω^c].

Remark 1.3.4. Martingales are another way to link harmonic functions to Brownian motion! Suppose u is a harmonic function on $\mathbb{D}$. Then, using the mean value property of u, one can check that $u(W_t)$ is a (bounded and continuous) martingale. By the martingale convergence theorem there is a random variable X such that $u(W_t) \overset{a.s.}{\to} X$. This means that for a.e Brownian trajectory starting at 0, the limit of u along the trajectory exists. In particular, for a.e point on $\mathbb{T}$, there is at least one trajectory converging to it, along which the limit of u exists. This (applied to $\text{Re} f$ and $\text{Im} f$) is precisely what we needed to complete the proof of Theorem 1.1.3 (see the remark following that theorem)!

1.4. Harmonic measure

Definition 1.4.1. Let $\Omega \subseteq \mathbb{C}$ be a region such that Ω^c has a connected component containing more than one point. Then $\tau := \inf\{t : B_t \notin \Omega\}$ is finite w.p.1. Define the $\mu_z(A) = \mathbf{P}_z(W_\tau \in A)$ for Borel set A. We call μ_z the *harmonic measure on* $\partial\Omega$ *in* Ω *as seen from* z. It is a probability measure supported on $\partial\Omega$.

An alternative definition is to define for each $z \in \Omega$ the linear functional T_z on $C_b(\partial\Omega)$ by $T_z f = u(z)$ where $f \in C_b(\partial\Omega)$ and u is the unique solution to the Dirichlet problem with boundary value f (the solution is unique for every bounded continuous f, by part (3) of Theorem 1.3.2). Then T_z is a positive linear functional with $T_z\mathbf{1} = 1$, whence by Riesz representation theorem,[2] there is a unique Borel probability measure μ_z on $\partial\Omega$ such that $T_z f = \int f\, d\mu_z$ for all $f \in C_b(\partial\Omega)$. That this definition agrees with the earlier one follows immediately from Theorem 1.3.2.

[2]By the footnote on the previous page, $\partial\Omega$ is a compact subset of the sphere $\mathbb{C}\bigcup\{\infty\}$ and hence any continuous function f on $\partial\Omega$ is also bounded. Of course, then f must be continuous at ∞ also to start with.

Example 1.4.2. On $\mathbb{D}$, $\mu_z(dt) = P(z, e^{it})dt$ where $P(z, e^{it}) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-e^{it}|^2}$ is the Poisson kernel for the unit disk. On $\mathbb{H}$, $\mu_z(dt) = \frac{y}{(x-t)^2+y^2}\frac{dt}{\pi}$ where $z = x+iy$. On $\mathbb{D}_+$ can you find $\mu_z([-1,1])$?

These examples are atypical. Usually we cannot find μ_z exactly but only estimate it. The next section will introduce one such estimate which will be of great use to us.

1.5. Beurling's theorem

If K is a compact subset of the plane we let $K^* := \{|z| : z \in K\}$ be its *radial projection*.

Theorem 1.5.1 (Beurling's projection theorem and estimate)**.** *Let $K \subseteq \mathbb{D}$ be compact and let $V = \mathbb{D}\backslash K$ and let $V^* = \mathbb{D}\backslash K^*$.*

1. *For any $z \in \mathbb{D}$ we have $\mu_z^V(K) \geq \mu_{-|z|}^{V^*}(K^*)$.*
2. *If K is connected, $0 \in K$ and $K \cap \mathbb{T} \neq \emptyset$, then $\mu_z^V(\mathbb{T}) \leq c_0\sqrt{|z|}$.*

PROOF. (1) By rotating we may assume that $z = -|z|$. Let $L_\theta = \{xe^{i\theta} : x \in \mathbb{R}\}$. Apply Lemma 1.5.2 with the lines $L_{\pi/2}, L_{\pm\pi/4}, L_{\pm\pi/8}, \ldots$ successively to get sets $K_1 = \mathscr{R}_{\pi/2}(K)$, $K_2 = \mathscr{R}_{\pi/4}\circ\mathscr{R}_{-\pi/4}(K_1)$, $K_3 = \mathscr{R}_{\pi/8}\circ\mathscr{R}_{-\pi/8}(K_2)$ etc. with $\mu_z^{V_j}(K_j)$ increasing in j. Then, K_j is confined to the sector $\{-\pi 2^{-j} \leq \arg z \leq \pi 2^{-j}\}$. $\mu_z^{V_j}(K_j)$ being the probability for a BM started at z to hit K_j before exiting $\mathbb{D}$, it can be proved that it converges to $\mu_z^{V^*}(K^*)$.

(2) Clearly $K^* = [0,1]$. Although part (a) applies to compact subsets of the disk, in the form $\mu_z^V(\mathbb{T}) \leq \mu_z^{V^*}(\mathbb{T})$ it applies equally to $K \subseteq \overline{\mathbb{D}}$. Thus $\mu_z^V(K) \leq \mu_{-r}^{[0,1]}([0,1])$ and one can explicitly compute that $\mu_{-r}^{[0,1]}(\mathbb{T}) = \frac{4}{\pi}\arctan\sqrt{r} \leq c_0\sqrt{r}$.

■

Lemma 1.5.2. *Let $z \in \mathbb{D}$ and let K be a compact subset of $\mathbb{D}$. Let L be a line passing through 0 and let $\mathscr{R}_L$ denote reflection across L. Let K_+ be the part of K on the same side as z and let K_- be the part of K on the other side. Let $\tilde{K} := K_- \bigcup \mathscr{R}_L(K_+)$. Set $V = \mathbb{D}\backslash K$ and $\tilde{V} = \mathbb{D}\backslash\tilde{K}$. Then, $\mu_z^V(K) \leq \mu_z^{\tilde{V}}(\tilde{K})$.*

PROOF. [Bernt Øksendal] Without loss of generality, we assume $\operatorname{Im} z > 0$ so that $K_+ = K \cap \mathbb{D}_+$ and $K_- = K \cap \overline{\mathbb{D}}_-$. We just write A' for $\mathscr{R}_L(A)$. If $H := K_+ \bigcup (K_- \backslash K'_+)$, then $H \subseteq K$ while $H^* = K^*$. Thus, proving the lemma for H implies the lemma for K. Thus we may assume that $K'_+ \cap K_- = \emptyset$. In addition, one can show from properties of Brownian motion that if $K_n \uparrow K$, then $\mathbf{P}_z(W \text{ hits } K_n \text{ before } \mathbb{T})$ converges to $\mathbf{P}_z(W \text{ hits } K \text{ before } \mathbb{T})$ and similarly for decreasing limits. Thus we may assume that K_+ and K_- are compact and that the mutual distances between K'_+, K_- and $[-1,1]$ are strictly positive.

Let $A = K' = K'_+ \bigcup K'_-$ and $B = \tilde{K}' = K'_+ \bigcup K'_-$ and let $I = [-1,1]$. Start a BM at z and run it till it hits $\mathbb{T}\bigcup K$and successively record which of the sets A, I, K are hit. For example, $AIAK$ denotes the set of all paths that hit A first, then I, then A and then K. Then,

$$\mu_z^V(K) = \mathbf{P}_z(K) + \mathbf{P}_z(AK) + \mathbf{P}_z(IK) + \mathbf{P}_z(AIK) + \mathbf{P}_z(IAK) + \mathbf{P}_z(AIAK) + \ldots \tag{1.5.1}$$

The sum is over all strings in which A and I alternate a number of times and is terminated by K. Similarly, we can write $\mu_z^{\tilde{V}}(\tilde{K})$ by recording which of the sets B, I

and $\tilde{K}$ are hit. But now, B can only be followed by I, so we get

$$\mu_z^{\tilde{V}}(\tilde{K}) = \mathbf{P}_z(I\tilde{K}) + \mathbf{P}_z(BI\tilde{K}) + \mathbf{P}_z(IBI\tilde{K}) + \mathbf{P}_z(BIBI\tilde{K}) + \dots \tag{1.5.2}$$

We show that each term here is equal to one of the terms in (1.5.1). More precisely,

Claim : $\mathbf{P}_z(I\tilde{K}) = \mathbf{P}_z(IK)$, $\mathbf{P}_z(BI\tilde{K}) = \mathbf{P}_z(AIK)$, $\mathbf{P}_z(IBI\tilde{K}) = \mathbf{P}_z(IAIK)$, etc.

For clarity of explanation, we first show that $\mathbf{P}_z(I\tilde{K}) = \mathbf{P}_z(IK)$. Indeed, write $IK = IK_- \bigsqcup IK_+$ and $I\tilde{K} = IK_- \bigsqcup IK_+'$ with the obvious meaning for the new symbols. For any path $\omega \in IK_+$, let $\tau = \inf\{t : \omega_t \in I\}$ and define a new path $\tilde{\omega}(t) := \omega(t)\mathbf{1}_{t\le\tau} + \mathscr{R}(\omega(t))\mathbf{1}_{t>\tau}$. Then, $\tilde{\omega} \in IK_+'$. Thus, $\mathbf{P}_z(IK_+) = \mathbf{P}_z(IK_+')$, by the strong Markov property. That $\mathbf{P}_z(IK) = \mathbf{P}_z(I\tilde{K})$ follows.

More generally, say to show that $P_z(IBI\tilde{K}) = \mathbf{P}_z(IAIK)$, let $V = \mathbb{D}\setminus K$ and $U = V\setminus I$ and write

$$\mathbf{P}_z(AIAIK) = \int_A\int_I\int_A\int_I \mu_z^U(dw_1)\mu_{w_1}^U(dt_1)\mu_{t_1}^V(dw_2)\mu_{w_2}^U(dt_2)\mu_{t_2}^V(K)$$

and similarly

$$\mathbf{P}_z(BIBI\tilde{K}) = \int_B\int_I\int_B\int_I \mu_z^{\tilde{U}}(dw_1)\mu_{w_1}^{\tilde{U}}(dt_1)\mu_{t_1}^{\tilde{V}}(dw_2)\mu_{w_2}^{\tilde{U}}(dt_2)\mu_{t_2}^{\tilde{V}}(\tilde{K}).$$

By considering reflection as before, $\mu_{t_2}^{\tilde{V}}(\tilde{K}) = \mu_{t_2}^V(K)$ for any $t_2 \in I$. Further, as $B = \mathscr{R}(A)$, the integrals are also equal (make a change of variables from w_i to $\mathscr{R}(w_i)$). ■

Beurling's estimate is very useful to us later. Here is one example that we use later.

Corollary 1.5.3. *Let Ω be a s.c. domain (other than the whole plane). Then, for any $z \in \Omega$, if $d = dist(z, \Omega^c)$, then for any $x > 0$ we have*

$$\mathbf{P}_z(W \text{ travels at least distance } xd \text{ away from } z \text{ before exiting } \Omega) \le \frac{c_0}{\sqrt{x}}.$$

PROOF. Let $\zeta \in \partial\Omega$ with $|z-\zeta| = d$. Consider $B(\zeta, xd)$ with $K = \partial\Omega$. By assumption that Ω is s.c., it follows that K is connected and connects ζ to $\partial B(\zeta, xd)$ (if xd is small enough, which is all we need to consider). Apply Beurling's estimate. ■

A standard application of harmonic measure in complex analysis is to Phragmen-Lindelöf type theorems. This may be omitted and is included only for general interest.

Corollary 1.5.4. *Let f be an analytic function on a neighbourhood of $\Omega = \{0 \le \mathrm{Re}\, z \le 1\}$ and suppose that $|f(z)| \le M_R := \exp\{\exp\{CR^{1-\epsilon}\}\}$ for some $\epsilon > 0$. If $|f(z)| \le M$ on $\{\mathrm{Re}\, z = 0$ or $1\}$ and Then, $|f(z)| \le M$ for all $z \in \Omega$.*

PROOF. For $R > 0$, let $S_R := [0,1] \times [-R, R]$ and let $u(z)$ be a harmonic function on S_R whose boundary values are equal to $\log|f|$. Since $\log|f|$ is subharmonic, we get $\log|f(z)| \le u(z)$ for any $z \in S_R$. Fix z and let $\theta_z(R)$ denote the harmonic measure of the top and bottom edges of S_R, as seen from z. Thus, $\log|f(z)| \le u(z) \le \theta_z(R)\log M_R + \log M$, by the Brownian motion solution to the Dirichlet problem. We need an estimate for $\theta_z(R)$. Divide S_R into $\lfloor 2R \rfloor$ squares of side 1 and argue that $\theta_z(R) \le e^{-cR}$ for some $c > 0$. Letting $R \to \infty$ we get $|f(z)| \le M$. ■

1.6. Schwarz's lemma and its many variants

Schwarz's lemma is the well-known statement that a holomorphic $f : \mathbb{D} \to \mathbb{D}$ that fixes 0 satisfies $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all $z \in \mathbb{D}$ and strict inequalities hold everywhere unless f is one-one and onto the disk. In which case it is a linear fractional transformation $f(z) = (az+b)/(cz+d)$ for some $a,b,c,d \in \mathbb{C}$ with $|a|^2 - |b|^2 = 1$. Pick generalized this statement to say that for any holomorphic $f : \mathbb{D} \to \mathbb{D}$, we have

$$(1.6.1) \qquad \frac{|f(z) - f(w)|}{|1 - f(z)\overline{f(w)}|} \le \frac{|z-w|}{|1 - z\overline{w}|} \text{ and } \frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2} \text{ for any } z, w \in \mathbb{D}$$

with strict inequalities everywhere unless f is one-one and onto the disk. These inequalities follow from Schwarz's statement by pre-composing (and post-composing) f with linear fractional transformation that takes 0 to z (respectively, $f(z)$ to 0).

In the form given by Pick, Schwarz's lemma attains a deep geometric meaning. On the disk $\mathbb{D}$, define a Riemannian metric $(1-|z|^2)^{-2}(dx^2+dy^2)$ called the *hyperbolic metric*. This means that an curve $\gamma : [0,1] \to \mathbb{D}$ has length $\ell(\gamma) = \int_0^1 (1 - |\gamma_t|^2)^{-1} |\dot{\gamma}_t| dt$. The length minimizing curves are called *geodesics* and they may be seen to be exactly arcs of circles that intersect $\mathbb{T}$ at right angles (this includes diameters of $\mathbb{D}$ which are arcs of circles of infinite radius).

If we call each $z \in \mathbb{D}$ a *point* and each geodesic a *line*, then this system of points and lines satisfy the axioms of the *hyperbolic geometry* discovered by Bolyai, Gauss and Lobachevsky. The axioms are the same as that of Euclidean geometry, except for the parallel postulate, which is replaced by a statement that '*given a line and a point not on it, there exist at least two lines through the point that do not intersect the given line*'. We say that the disk with the hyperbolic metric forms a model for hyperbolic geometry.

Exercise 1.6.1. Show that $\mathbb{H}$ with the metric $y^{-2}(dx^2 + dy^2)$ is isometric to the hyperbolic metric on $\mathbb{D}$. What are the geodesics?

Returning to Schwarz-Pick lemma, the inequalities 1.6.1 state precisely that any holomorphic map from the disk to itself is a contraction of the hyperbolic metric. And if it is not strictly contracting even at one point, then it must be an isometry of the disk, in other words a linear fractional transformation that maps the disk injectively onto itself. Here are some variants of Schwarz's lemma that we shall need.

Lemma 1.6.2. (1) *Let $f : \mathbb{H} \to \mathbb{H}$ be holomorphic. Then,*

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \le \frac{1}{\operatorname{Im} z} \quad \textit{and} \quad \frac{\left| f(z) - f(w) \right|}{\left| f(z) - \overline{f(w)} \right|} \le \frac{|z-w|}{|z - \overline{w}|}$$

with strict inequalities unless $f(z) = (az+b)/(cz+d)$ for some $a,b,c,d \in \mathbb{R}$ with $ad - bc = 1$.

(2) *Let $f : \mathbb{D} \to \mathbb{D}$ be any holomorphic function that extends analytically to, and fixes two distinct points $a, b \in \mathbb{T}$. Then, $f'(a)f'(b) \ge 1$ with equality if and only if f is a hyperbolic isometry of $\mathbb{D}$.*

(3) *Let $f : \mathbb{H} \to \mathbb{H}$ be holomorphic function that extends analytically to, and fixes ∞. Suppose $f(z) = \alpha z + O(1)$ as $z \to \infty$. If f extends analytically to some $x \in \mathbb{R}$ and $f(x) \in \mathbb{R}$, then $|f'(x)| \ge \alpha$.*

PROOF. (1) Let $\varphi(z) = \frac{z-i}{z+i}$ and apply Schwarz-Pick to $\varphi \circ f \circ \varphi^{-1}$.

(2) Derive this one from part (3) using a linear fractional transformation that maps a and b to 0 and ∞. We have not omitted absolute values by mistake. Both $f'(a)$ and $f'(b)$ must be positive as f maps $\mathbb{D}\cap\mathbb{D}(a,\delta)$ and $\mathbb{D}\cap\mathbb{D}(b,\delta)$ into $\mathbb{D}$.

(3) α must be positive as f maps $\mathbb{H}$ into $\mathbb{H}$. Without loss of generality take $x=0=f(x)$ and write $f(z)=\beta z+O(z^2)$ as $z\to 0$ so that $f'(0)=\beta$ which is also positive as f maps $\mathbb{H}$ into $\mathbb{H}$ near 0. Now take $z=iy$ and $w=iv$ and apply part (1) to get

$$\frac{\left|\, i\beta y - i\alpha v + O(1)\,\right|}{\left|\, i\beta y + i\alpha v + O(1)\,\right|} \le \frac{|iy-iv+O(1)|}{|iy+iv+O(1)|}.$$

The left hand side is $1-2\frac{\beta}{\alpha}\frac{y}{v}+O(y^2/v^2)$ while the right hand side is $1-2\frac{y}{v}+O(y^2/v^2)$ from which it follows that $\alpha\le\beta$ as claimed. ∎

1.7. Half-plane capacity

Definition 1.7.1. For a hull K, Theorem 1.2.3 yields a unique conformal map $\Phi_K: H\backslash K\to\mathbb{H}$ such that $\Phi_K(z)=z+\frac{a(K)}{z}+O(z^{-2})$ as $z\to\infty$. The number $a(K)$ is called the *half-plane capacity* of K as seen from infinity.

Example 1.7.2. If $K=r\mathbb{D}_+$, then $\Phi_K(z)=z+\frac{r^2}{z}+O(z^{-2})$, hence $a(K)=r^2$. If $K=[0,it]$, then $\Phi_K(z)=\sqrt{z^2+t^2}=z+\frac{t^2}{2z}+O(z^{-2})$ whence $a(K)=t^2/2$.

Lemma 1.7.3. *We have the following properties of half-plane capacity.*

(1) *If $K_1\subseteq K_2$ are hulls, then $a(K_1)\le a(K_2)$ with equality if and only if $K_1=K_2$.*
(2) *$a(rK+t)=r^2a(K)$ if $r\ge 0$ and $t\in\mathbb{R}$.*
(3) *Let $\tau_K:=\inf\{t:W_t\in K\bigcup\mathbb{R}\}$. Then, $a(K)=\lim_{y\to\infty} y\mathbf{E}_{iy}\left[\operatorname{Im}W_{\tau_K}\right]$.*

PROOF. (1) We saw that $a(K)\ge 0$ with equality if and only if $K=\emptyset$. Now, let $K_3:=\Phi_{K_1}(K_2\backslash K_1)$. Then, $\Phi_{K_2}=\Phi_{K_3}\circ\Phi_{K_1}$ from which it follows that $a(K_2)=a(K_1)+a(K_3)\ge a(K_1)$ with equality if and only if $K_3=\emptyset$ which is equivalent to $K_1=K_2$.

(2) True, since $\Phi_{rK+t}(z)=r\Phi_K\left(\frac{z-t}{r}\right)+t=(z-t)+\frac{r^2a(K)}{z-t}+O((z-t)^{-2})+t=z+\frac{r^2a(K)}{z}+O(z^{-2})$.

(3) $\operatorname{Im}\{\Phi_K(z)-z\}$ is a bounded harmonic function on $\mathbb{H}\backslash K$ and equal to $-\operatorname{Im}z$ for $z\in K\bigcup\mathbb{R}$. By Theorem 1.3.2 we have $\mathbf{E}_z\left[\operatorname{Im}W_{\tau_K}\right]=\operatorname{Im}z-\operatorname{Im}\Phi_K(z)$. Set $z=iy$, multiply by y to get $y\mathbf{E}_{iy}\left[\operatorname{Im}W_{\tau_K}\right]=y^2-y\left(y-\frac{a(K)}{y}+O(y^{-2})\right)\to a(K)$ as $y\to\infty$. ∎

Exercise 1.7.4. If K is a hull with $r(K)<1$, then $a(K)=\frac{2}{\pi}\int_0^\pi \mathbf{E}_{e^{it}}\left[\operatorname{Im}B_\tau\right]\sin t\,dt$.

With the probabilistic interpretation, we can estimate $a(K)$ using Brownian motion hitting probabilities which in turn are estimated by Beurling's estimate. Thus we have the following fact.

Theorem 1.7.5. *Let K_1, K_2 be two hulls.*

(1) *If $K_1\subseteq K_2$ and every point of $\partial(\mathbb{H}\backslash K_2)$ (which may be smaller than $K_2\bigcup\mathbb{R}$) is within δ distance of $K_1\bigcup\mathbb{R}$. Then $a(K_1)\le a(K_2)\le a(K_1)+c_0\sqrt{\delta}\,r(K_2)^{3/2}$.*
(2) *K_1,K_2 general with $r(K_i)\le r$. If every point of $\partial(\mathbb{H}\backslash K_2)$ is within δ distance of $K_1\bigcup\mathbb{R}$ and* vice versa*, then $|a(K_1)-a(K_2)|\le c_0\sqrt{\delta}\,r^{3/2}$.*

PROOF. (1) We know that $a(K_1) \le a(K_2)$. To show the second inequality, start Brownian motion W at iy and let $\tau \le \tau_2 \le \tau_1$ be the hitting times of $(2r_2\mathbb{T})\bigcup\mathbb{R}$, of $K_2\bigcup\mathbb{R}$ and of $K_1\bigcup\mathbb{R}$, respectively.

Let $A := \{|W_\tau| = 2r_2\}$. If A^c occurs, then $W_\tau \in \mathbb{R}$ and $\tau_1 = \tau_2 = \tau$ and hence $\operatorname{Im} W_{\tau_2} - \operatorname{Im} W_{\tau_1} = 0$. On the event $A := \{|W_\tau| = 2r_2\}$, we may deduce from corollary 1.5.3 that $\mathbf{P}\left(\operatorname{Im} W_{\tau_2} - \operatorname{Im} W_{\tau_1} > x|A\right) \le c_0\sqrt{\delta/x}$ for $x < r_2$ and equals 0 for $x > r_2$. This is because W_{τ_2} is within δ distance of some point in $K_1\bigcup\mathbb{R}$.

Further, $\mathbf{P}_{iy}(A) \sim \frac{r_2}{\pi y}$ as $y \to \infty$, hence using the probabilistic interpretation of half-plane capacity, $a(K_2) - a(K_1)$ is the limit as $y \to \infty$ of

$$\begin{aligned} y\mathbf{E}_{iy}\left[\operatorname{Im} W_{\tau_2}\right] - y\mathbf{E}_{iy}\left[\operatorname{Im} W_{\tau_1}\right] &= y\mathbf{P}_{iy}(A)\,\mathbf{E}_{iy}\left[\operatorname{Im} W_{\tau_2} - \operatorname{Im} W_{\tau_1}\,\middle|\,A\right] \\ &\le c_0 r_2 \int_0^{r_2} \frac{\sqrt{\delta}}{\sqrt{x}}dx = c_0\delta^{1/2}r_2^{3/2}. \end{aligned}$$

(2) Immediate upon applying the first part to the pairs $K_1 \subseteq K_1\bigcup K_2$ and $K_2 \subseteq K_1\bigcup K_2$. ■

Example 1.7.6. Let $K_t = \{e^{i\theta} : 0 < \theta \le \pi t\}$ for $t < 1$ and $K_1 = \mathbb{D}_+ := \{z \in \mathbb{H} : |z| \le 1\}$. Then, $a(K_t)$ is continuous in t, including $t = 1$. Had our condition in part (1) of Theorem 1.7.5 said $K_2\bigcup\mathbb{R}$ instead of $\partial(\mathbb{H}\backslash K_2)$, continuity at $t = 1$ would not have followed!

Suppose γ is a simple curve in $\overline{\mathbb{H}}$ with $\gamma_0 \in \mathbb{R}$. Then, $\mathbb{H}\backslash\gamma[0,t]$ has a unique unbounded component whose complement we denote by K_t. Then K_t is an increasing family of hulls. Again, $a(K_t)$ is continuous in t. This too follows easily from Theorem 1.7.5 because $\partial(\mathbb{H}\backslash K_t) \subseteq \gamma[0,t]$ and γ is continuous. A special case is when γ is a simple curve with $\gamma_t \notin \mathbb{R}$ for any $t > 0$ and $\gamma_0 \in \mathbb{R}$. In this case, $K_t = \gamma[0,t]$.

Remark 1.7.7. There is a well-known quantity called logarithmic capacity. There are many ways to define it, but for a connected compact set K, its capacity $c(K)$ turns out to be equal to $|\varphi'(\infty)|$ where φ is a Riemann map from $\overline{\mathbb{C}}\backslash K$ onto $\Delta := \mathbb{C}\backslash\mathbb{D}$ with $\varphi(\infty) = \infty$. Here $\varphi'(\infty)$ is the reciprocal of the coefficient a in the expansion $\varphi(z) = az + b + cz^{-1} + \dots$ near ∞.

If K is a hull in the upper half-plane, is $a(K)$ related to $c(K)$ or perhaps $c(K\bigcup\overline{K})$? If Ψ maps $\mathbb{H}\backslash K$ onto $\mathbb{H}\backslash\mathbb{D}_+$, then it extends to a map from $\mathbb{H}\backslash(K\bigcup\overline{K})$ onto Δ. The expansion at ∞ may be written as $\Psi_K(z) = az+b+cz^{-1}+\dots$. Then $\Phi_K(z) = a^{-1}(\Psi(z)+\Psi(z)^{-1}) - b/a$. This has the expansion $\Phi_K(z) = z + a^{-1}(c+a^{-1})z^{-1} + O(z^{-2})$ which shows that $a(K) = a^{-1}c + a^{-2}$. Of course $c(K\bigcup\overline{K}) = a^{-1}$. There does not seem to be any simple relationship between $a(K)$ and $c(K\bigcup\overline{K})$.

1.8. A heuristic derivation of Löwner's equation

Let γ_t be a simple curve with $\gamma(0) \in \mathbb{R}$ and $\gamma(t) \in \mathbb{H}$ if $t > 0$. Let g_t be conformal maps from $\mathbb{H}\backslash\gamma[0,t] \to \mathbb{H}$. We want to understand how the maps g_t vary with t, the ultimate goal being to derive Löwner's differential equation

$$\dot{g}_t(z) = \frac{\dot{a}_t}{g_t(z) - U_t}$$

where $a_t = a(\gamma[0,t])$, $U_t = g_t(\gamma_t)$ and $\dot{a}_t, \dot{g}_t$ denote time (t) derivatives. Among several issues raised by this statement, we have addressed a few. For example U_t is well-defined because γ_t corresponds to only one prime end in $\mathbb{H}\backslash\gamma[0,t]$. What about differentiability of a_t? Theorem 1.7.3 shows that a_t is strictly increasing and Theorem 1.7.5 shows that a_t is continuous. Therefore, γ can always be reparameterized so that a_t is smooth. In particular, if we set $\eta_s = \gamma_{a^{-1}(2s)}$, defined for $s < a(\infty)$, then $a(\eta[0,s]) = 2s$. η is called the *natural parameterization* of γ.

Assume now that $\gamma(0)$ is in its natural parameterization, that is $a_t = 2t$. Fix t and let $\Gamma_s = g_t(\gamma_{t+s})$. If $G_s = \Phi_{\Gamma[0,s]}$, then for any $h > 0$we have $g_{t+h} = G_h \circ g_t$. From this, it is clear that Γ is also a simple curve in its natural parametrization (see the proof of the first part of Lemma 1.7.3). If we heuristically replace $\Gamma[0,h]$ by a vertical slit $[U_t, U_t + i2\sqrt{h}]$ having the same half-plane capacity $2h$, then by Example 1.2.5, we get $g_{t+h}(z) \approx \sqrt{(g_t(z) - U_t)^2 + 4h}$. Differentiate by h and set $h = 0$ to get

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}$$

which is exactly Löwner's equation! If you have been able to calculate the conformal maps for oblique slits asked for in Example 1.2.5, then check that replacing $\Gamma[0,h]$ by any oblique slit with the same half-plane capacity $2h$ will also result in the same differential equation.

The gaps in this heuristic are that we only calculated the right derivative of g_t above and that we did not justify how to replace $\Gamma[0,h]$ by a slit. Both these issues are addressed by proving several estimates in the next section about how conformal maps Φ_K differ if the hulls K differ only a little. Then we put together the proof in section 1.10.

1.9. Closeness of conformal maps

We develop some required estimates in this section. The first theorem makes precise in what sense the approximations $\Phi_K(z) \approx z$ or $\Phi_K(z) \approx z + a(K)z^{-1}$ hold. The last one gets an estimate for how much a set near K can be blown up by Φ_K.

Theorem 1.9.1. *Let K be a hull and Φ_K as before. Then, $|\Phi_K(z) - z| \le 3r(K)$ for all $z \in \mathbb{H}\setminus K$.*

PROOF. By scaling it suffices to assume that $r(K) < 1$ and prove that $|\Phi_K(z) - z| \le 3$ for all $z \in \mathbb{H}\setminus K$. For any fixed $x \ge 1$ (or $x \le -1$) we claim that $\Phi_K(x)$ is increasing (respectively, decreasing) in K, as long as $r(K) < 1$. Assuming this claim, apply it to the hulls $\emptyset \subseteq K \subseteq \overline{\mathbb{D}}_+$ to get

$$x \le \Phi_K(x) \le x + \frac{1}{x} \text{ for } x \ge 1 \text{ and } x \ge \Phi_K(x) \ge x + \frac{1}{x} \text{ for } x \le -1.$$

In particular, $|\Phi_K(x) - x| \le 3$ for $|x| \ge 1$. For $z \in \overline{\mathbb{D}}_+\setminus K$, we must have $\Phi_K(z) \in [\Phi_K(-1), \Phi_K(1)] \subseteq [-2, 2]$. Thus $|z - \Phi_K(z)| \le 3$. Thus we have shown that $\limsup |\Phi_K(z) - z| \le 3$ whenever $z \to \partial(\mathbb{H}\setminus K)$ (note that the limit value is 0 as $z \to \infty$). By maximum modulus principle, $|\Phi_K(z) - z| \le 3$ for any $z \in \mathbb{H}\setminus K$.

It remains to prove the claim that $\Phi_K(x) \le \Phi_L(x)$ for $x \ge 1$ if $K \subseteq L$ with $r(L) < 1$. If $J = \Phi_K(L\setminus K)$, then $\Phi_L = \Phi_J \circ \Phi_K$. Hence it suffices to show that $\Phi_K(x) \ge x$ for a single hull K. The map $\Phi_K(z) - \Phi_K(x) + x$ fixes x and ∞ and has derivative 1 at ∞. Applying part (3) of Lemma 1.6.2 to Φ_K^{-1} we get $\Phi_K'(x) \le 1$. For large y, $\Phi_K(y) = y + a(K)y^{-1} + O(y^{-2}) > y$. Therefore $\Phi_K(x) > x$ for all $x > 1$. ■

Theorem 1.9.2. *If K is a hull, then $\left|\Phi_K(z) - z - \frac{a(K)}{z}\right| \le c_0 \frac{a(K)r(K)}{|z|^2}$ whenever $|z| > 16r(K)$.*

The proof we give is exactly as in Lawler. Later we describe an attempt at a slightly different reasoning that did not quite work. Do let me know if you see how to fix it.

PROOF. Scale and assume that $r(K) = 1$. Let $g(z) = \Phi_K(z) - z - \frac{a}{K} z$ and let $v = \operatorname{Im} g$. By harmonicity of v and Theorem 1.3.2, we write

$$v(z) = \mathbf{E}_z[\operatorname{Im}\{\Phi_K(W_\tau) - W_\tau\}] - a(K)\operatorname{Im}(1/z) = -\mathbf{E}_z[\operatorname{Im} W_\tau] - a(K)\operatorname{Im}(1/z)$$

where τ is the hitting time of $K \bigcup \mathbb{R}$ by the Brownian motion W. If $\hat{\tau}$ denote the hitting time of $\mathbb{T} \bigcup \mathbb{R}$, then by the strong Markov property, we can write

$$\begin{aligned} v(z) &= -\mathbf{E}_z\left[\mathbf{E}_{e^{it}}[\operatorname{Im} W_\tau]\right] - a(K)\operatorname{Im}(1/z) \\ &= \int_0^\pi \mathbf{E}_{e^{it}}[\operatorname{Im} W_\tau]\left(p(z, e^{it}) - \frac{2}{\pi}\sin t\right) dt \end{aligned}$$

where $p(z, e^{it})$ is the density of $W_{\hat{\tau}}$ on $\mathbb{T}$ and by writing $a(K)$ as in Exercise 1.7.4. In Lemma lem:hittingofcircle we calculate $p(z, e^{it})$ explicitly and show that $p(z, e^{it}) = \operatorname{Im}(1/z)\frac{2}{\pi}\sin t(1 + O(1/|z|))$. Hence, we get

$$|v(z)| \le c_0 \operatorname{Im}(1/z)\frac{1}{|z|}\int_0^{2\pi} \mathbf{E}_{e^{it}}[\operatorname{Im} W_\tau]\frac{2}{\pi}\sin t\, dt \le c_0 a(K)\frac{\operatorname{Im}(z)}{|z|^3}.$$

.

Now, consider $|w| \geq 2$. Then the disk $\mathbb{D}(w, |w|/2)$ is contained in $\{|z| > 1\}$ and hence, we can use Poisson integral formula to write for $|w' - w| < R := |w|/2$,

$$v(w') = \int_0^{2\pi} v(w + Re^{it}) \frac{R^2 - |w'|^2}{|w' - Re^{it}|^2} \frac{dt}{2\pi}$$

from which we differentiate with respect to x (or y) and set $w' = w$ and use the bounds on v (that holds for all t since $|w + Re^{it}| \geq |w|/2 \geq 1$). We get

$$\max\left\{|v_x(w)|, |v_y(w)|\right\} \leq c_0 a(K) \frac{1}{R} \frac{\operatorname{Im} w}{|w|^3} \leq c_0 \frac{1}{|w|^3}.$$

Since $v = \operatorname{Im} g$, we get $|g'(w)| = |v_y(w) + iv_x(w)| \leq c_0 a(K) \frac{1}{|w|^3}$. Note that $g(\infty) = 0$. Therefore, integrating g' along a path from z to ∞ we get

$$|g(z)| \leq \int_0^\infty |g'(z + it)| dt \leq c_0 a(K) \int_0^\infty \frac{|z + it|^3}{d} t \leq c_0 \frac{a(K)}{|z|^2}.$$

This completes the proof. ■

We used the following lemma in the proof.

Lemma 1.9.3. *Let $|z| > 1$ and $\tau := \inf\{t : W_t \in \mathbb{T} \bigcup \mathbb{R}\}$. Let $p(z, e^{it})$ be the density of W_τ conditional on $W_\tau \in \mathbb{T}$. Then, $p(z, e^{it}) = \operatorname{Im}(1/z)\frac{2}{\pi} \sin t (1 + O(1/|z|))$ for all $t \in [0, \pi]$.*

PROOF. Let $|z| > 1$. For any arc $I \subset \mathbb{T}$, the harmonic measure $\mu_z(I)$ is the value at z of the harmonic function which has boundary values 1 on I and 0 on $\mathbb{R} \bigcup \{\infty\} \bigcup (\mathbb{T} \setminus I)$. By conformal invariance of harmonic functions, we can map it forward by $z \to z + 1/z$ which takes $\mathbb{T}$ to $[-2, 2]$. By knowledge of the Poisson kernel on $\mathbb{H}$, the hitting density of W, started from $w = z + 1/z$ is given by $p(s) = \frac{\operatorname{Im} w}{\pi |w - s|^2}$ for $s \in \mathbb{R}$. Pulling it back to the original problem, we find that

$$p(z, e^{it}) = \frac{\operatorname{Im}(z + z^{-1})}{\left|z + z^{-1} - 2\cos t\right|^2} \frac{2 \sin t}{\pi}$$

which can easily be seen to be equal to $\operatorname{Im}(1/z)\frac{2}{\pi} \sin t (1 + O(1/|z|))$. ■

Remark 1.9.4. Taking $z = iy$ in Lemma1.9.3 and letting $y \nearrow \infty$, in conjunction with part (3) of Lemma 1.7.3 solves Exercise 1.7.4

Theorem 1.9.5. *Let K be a hull and let $\eta : [0, 1] \to \overline{\mathbb{H}}$ be a curve such that $\eta(0) \in K \bigcup \mathbb{R}$ and $\eta(t) \in \mathbb{H} \setminus K$ for $t \in (0, 1]$. Let d_η be the diameter of η and let $\ell_\eta := \sup_t \operatorname{Im} \eta_t$. Assume that $d_\eta \leq 10\ell_\eta$. Then, there exists a constant c_0 such that dia$\left(\Phi_K(\eta)\right) \leq c_0 \sqrt{d_\eta \cdot \ell_\eta}$.*

The main idea in the proof is to replace diameter by a conformally invariant quantity that is comparable to the diameter of a set. The conformally invariant quantity we use will be the harmonic measure. For instance, the length of an interval on the line can be estimated by the hitting probability by a Brownian motion started far off, that is, if $\tau = \inf\{t : W_t \in \mathbb{R}\}$, then $\lim_{y \to \infty} y \mathbf{P}_{iy}(W_\tau \in [a, b]) = \frac{1}{\pi}(b - a)$. The following exercise provides the needed generalization of this.

Exercise 1.9.6. Then there exists c_1, c_2 such that for any connected hull K we have

$$c_1 \operatorname{dia}(K) \leq \lim_{y \to \infty} y \mathbf{P}_{iy}(W_\tau \in [a, b]) \leq c_2 \operatorname{dia}(K)$$

where τ is the hitting time of $K \bigcup \mathbb{R}$ by W.

PROOF. [Proof of Theorem 1.9.5] The idea is as follows. Let W be a Brownian motion started at iy for a large y and let τ be the hitting time of $K\bigcup\mathbb{R}$. We use an argument (analogous to what is needed for the upper bound in Exercise 1.9.6) to bound the probability that W hits η before time τ. This probability is the harmonic measure of η as seen from iy in $\mathbb{H}\backslash K$. By conformal invariance of harmonic functions, it is the same as the harmonic measure of $\Phi_K(\eta)$ as seen from $\Phi_K(iy)$ in $\mathbb{H}$. Then using the lower bound in Exercise 1.9.6, we get the desired bound for $\text{dia}(\Phi_K(\eta))$.

Now for details. Let $L = K\bigcup\mathbb{D}(\eta_0, 10\ell_\eta)$ and let $\tilde{\tau}$ be the hitting time of $L\bigcup\mathbb{R}$ by W. It s easy to prove that $\mathbf{P}_{iy}(W_{\tilde{\tau}} \in \eta_0 + \ell_\eta\mathbb{T}) \le c_0 y^{-1}\ell_\eta$ (left as exercise). If $W_{\tilde{\tau}} \notin \eta_0 + \ell_\eta\mathbb{T}$, then there is no hope for W to hit η before time τ. On the other hand, if $W_{\tilde{\tau}} = \zeta \in \eta_0 + \ell_\eta\mathbb{T}$ then we claim that $P_\zeta(W \text{ hits } \eta \text{ before time } \tau) \le c_0\sqrt{d_\eta/\ell_\eta}$. This follows by Beurling's estimate but with a small twist. See Exercise 1.9.7.

Puting everything together, we get $yP_iy(W \text{ hits } \eta \text{ before time } \tau) \le c_0\sqrt{d_\eta\ell_\eta}$. We get the same bound for $yP_{\Phi_K(iy)}(W \text{ hits } \Phi_K(\eta) \text{ before time } \tau)$ by conformal invarince of harmonic functions. Since $\Phi_K(iy) = y + O(y^{-1})$, using the lower bound in Exercise 1.9.6 we get $\text{dia}(\Phi_K(\eta)) \le c_0\sqrt{d_\eta\ell_\eta}$. ∎

Exercise 1.9.7. Let K be a connected set in the plane connecting 0 to $\mathbb{T}$. Then for any $\zeta \in \mathbb{T}$, we have $P_\zeta(W \text{ hits } \epsilon\mathbb{T} \text{ before hitting } K) \le c_0\sqrt{\epsilon}$.

1.10. Chordal version of Löwner's differential equation

Let γ be a simple curve in $\overline{\mathbb{H}}$ with $\gamma_0 = 0$ and $\gamma_t \in \mathbb{H}$ for $t > 0$. Let $g_t : \mathbb{H}\backslash\gamma[0,t] \to \mathbb{H}$ be the unique conformal map such that $g_t(z) = z + \frac{a_t}{z} + O(z^{-2})$ as $z \to \infty$. Here $a_t = a(\gamma[0,t])$ is the half-plane capacity. Then, by Theorem 1.7.5 a_t continuous. Lemma 1.7.3 asserts that it is also strictly increasing, as γ is simple. Thus, there is a unique way to reparameterize γ so that $a_t = 2t$ for all t. We call this the *standard parameterization*.

Theorem 1.10.1 (Löwner)**.** *Let γ be a simple curve in standard parameterization and g_t the associated conformal maps normalized hydrodynamically. Then, $U_t := g_t(\gamma_t)$ is continuous in t and $\partial_t g_t(z) = \frac{2}{g_t(z)-U_t}$ for $z \in \mathbb{H}\backslash\gamma[0,t]$.*

PROOF. Let $K_t = \gamma[0,t]$. Fix $T > 0$ and consider $t < s \in [0,T]$ throughout. Let $h_{t,s}(z) = g_s \circ g_t^{-1}$. In earlier notation $g_t = \Phi_{K_t}$ and $h_{t,s} = \Phi_{K_{t,s}}$ where $K_{t,s} := g_t(K_s\backslash K_t)$ (we translate the hull so that it is close to the origin). Then,

$$\begin{aligned}
\|g_t - g_s\|_{\sup_{\mathbb{H}\backslash K_s}} &= \sup\left\{|h_{t,s}(z) - z| : z \in K_{t,s}\right\} \\
&\le 3r(K_{t,s} - U_t) \qquad \text{(by Theorem 1.9.2 applied to } K_{t,s} - U_t) \\
&\le C_{\gamma,T}\sqrt{\text{dia}\gamma([t,s])} \qquad \text{(by Theorem 1.9.5).}
\end{aligned}$$

From this it immediately follows that $t \to U_t$ is continuous (see Remark 1.10.2).

Next, we observe that if $z \in \mathbb{H}\backslash K_s$ then

$$\begin{aligned}
\left|g_s(z) - g_t(z) - \frac{2(s-t)}{g_t(z)-U_t}\right| &= \left|h_{t,s}(w) - w - \frac{2(s-t)}{w-U_t}\right| \qquad \text{(where } w = g_t(z)) \\
&\le C_{\gamma,T}\frac{(s-t)\text{dia}(K_{t,s})}{|g_t(z)-U_t|^2} \qquad (1.10.1)
\end{aligned}$$

where the last inequality is implied by Theorem 1.9.2 if $|g_t(z) - U_t| > 2r(K_{t,s})$.

If we fix and $z \in \mathbb{H}\backslash K_s$ and let $t \uparrow s$, then the last condition holds for t close enough to s. Divide (1.10.1) by $t-s$ and let $t \uparrow s$ to get $\dot{g}_{s-}(z) = \frac{2}{g_s(z)-U_s}$. Here we used the continuity of $g_t(z)$ and U_t in t. Similarly, we can fix $z \in \mathbb{H}\backslash K_t$, divide by $s-t$ and let $s \downarrow t$ in (1.10.1). The condition $|g_t(z)-U_t| > 2r(K_{t,s})$ holds again for s close enough to t and we get $\dot{g}_{t+}(z) = \frac{2}{g_t(z)-U_t}$. Thus we have proved Löwner's differential equation $\dot{g}_t(z) = \frac{2}{g_t(z)-U_t}$. ∎

Remark 1.10.2. In the course of the proof we claimed that the continuity of U_t in t follows from $\|g_t - g_s\| \le C_{\gamma,T} \sqrt{\mathrm{dia}\gamma([t,s])}$. As a matter of fact, this inequality can be used to prove the existence of $U_t := \lim g_t(z)$ as $z \to \gamma_t$ in $\mathbb{H}\backslash K_t$ without having to invoke the theorem on boundary values of conformal maps. We leave this as an exercise or to look up the proof of Lemma 4.2 in Lawler's book.

Remark 1.10.3. Suppose K_t is a one-parameter family of hulls such that $K_t \subsetneq K_s$ for $t < s$ and let g_t be the hydrodynamically normalized conformal maps from $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$. Assume in addition that $\bigcap_{\delta>0} K_{t,t+\delta} = \{U_t\}$ for every t. We then say that the family of hulls is right continuous. It is clear that a right continuous family of hulls can be reparameterized so that $a_t := a(K_t) = 2t$. Then the proof of Theorem 1.10.1 goes through exactly as stated above. In particular, U_t is continuous in t and $\dot{g}_t(z) = \frac{2}{g_t(z)-U_t}$ for any $z \in \mathbb{H}\backslash K_t$ and any t.

U is called the *driving function*.

1.11. Continuously growing hulls

Suppose K_t is a one-parameter family of hulls such that $K_t \subsetneq K_s$ for $t < s$ and let g_t be the hydrodynamically normalized conformal maps from $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$. Define $K_{t,s}$ as in the proof above, and *assume* that for every t there is a point $U_t \in \mathbb{R}$ such that $\bigcap_{\delta>0} \overline{K}_{t,t+\delta} = \{U_t\}$. We then say that the family of hulls is right continuous with *driving function* U.

From $a(K_s) = a(K_t) + a(K_{t,s})$ and Theorem 1.7.5 we see that $a(K_t)$ is continuous. Hence a right continuous family of hulls can be reparameterized so that $a_t := a(K_t) = 2t$.

In this case, one half of the proof of Theorem 1.10.1 goes through exactly as stated above.

Proposition 1.11.1. *Let $K_\bullet$ be right-continuous with driving function U. Assume that the K is in its natural parameterization. Then U_t is* right *continuous in t and $\dot{g}_{t+}(z) = \frac{2}{g_t(z)-U_t}$ for any $z \in \mathbb{H}\backslash K_t$ and any t.*

PROOF. As usual it suffices to check this at $t = 0$ with $U_0 = 0$. For $h > 0$ and $z \in K_{2h}$ we have $|g_h(z)| \le |g_h(z) - z| + |z|$ which is bounded by $3r(K_h) + r(K_{2h})$ which goes to zero as $h \to 0$. Since $U_0 = 0$ and $U_h \in K_{h,2h} = g_h(K_{2h})$, it follows that $U_h - U_0 \to 0$ as $h \downarrow 0$. This shows right continuity of U. The right derivative of g_t can be found exactly as before. ∎

Exercise 1.11.2. Let γ be a curve in $\overline{\mathbb{H}}$ and let K_t be the hull generated by $\gamma[0,t]$, that is $\mathbb{H}\backslash K_t$ is the unique unbounded component of $\mathbb{H}\backslash\gamma[0,t]$. Assume that $K_t \neq K_s$ for any $t < s$.[3] Then K_t is strictly increasing and right continuous.

The left continuity of U and the left derivative of g_t don't follow automatically. For example, if γ as in the above exercise crosses itself, the U jumps. To be specific,

[3]If K_t is an increasing family of hulls, it can be reparameterized so that it is strictly increasing. But if $K_\bullet$ is also generated b a curve γ, then it may not be possible to reparameterize γ so that K_t is strictly increasing in t.

take $\gamma_t = 2it$ for $t \le 1$ and $\gamma_t = ie^{i(t-1)}$ for $1 \le t \le 1+2\pi$. At $t_0 = 1+\pi$, the curve crosses itself. There are two prime ends corresponding to the boundary point 1 of $\mathbb{H}\setminus K_{1+\pi}$ and U_{t_0} and U_{t_0-} are the limits of $g_{t_0}(z)$ as $z \to 1$ along these two prime ends.

Definition 1.11.3. If K is right continuous with driving function U, and $t \to U_t$ is continuous, then we say that K is a *continuously growing family of hulls*.

Since this definition puts in exactly what is needed to make the proof of Theorem 1.10.1, we get

Proposition 1.11.4. *Let $K_\bullet$ be a continuously growing family of hulls with driving function U. Assume that the K is in its natural parameterization. Then $\dot{g}_t(z) = \frac{2}{g_t(z)-U_t}$ for any $z \in \mathbb{H}\setminus K_t$ and any t.*

1.12. Löwner evolution in reverse

We start with a converse to Proposition 1.11.4.

Theorem 1.12.1. *Let U_t be a continuous real valued function with $U_0 = 0$. Then there exists a unique continuously increasing family of hulls $K_\bullet$ that is naturally parameterized and whose driving function is U.*

PROOF. Consider the differential equation $\dot{x}_t = 2/(x_t - U_t)$ in the complex plane. For each z, let T_z be the maximal time for which the solution exists, starting with the initial condition $x_0 = z$. This unique solution we denote by $g_t(z)$. Thus $g_0(z) = z$ and $\partial_t g_t(z) = \frac{2}{g_t(z)-U_t}$ for $t \in [0, T_z)$. Clearly, $T_z = \inf\{t : g_t(z) = U_t\}$.

Let $K_t := \{z \in \mathbb{H} : T_z \le t\}$. By continuity of solutions in the initial conditions, K_t is closed in $\mathbb{H}$. If $\|U\|_t = \max\{U_s : s \le t\}$, then the bound on the speed $|\dot{g}_t(z)| = 2/\|U\|_t$ implies that K_t is bounded. Further, if $z \in K_t$, then $g_s(z) \in K_t$ for any $s < t$. Since $g_t(z) = U_t$, this shows that $K_t \bigcup \mathbb{R}$ is connected. Hence $\mathbb{H}\setminus K_t$ is simply connected. Therefore K_t are hulls, and obviously (weakly) increasing.

Now we claim that g_t is a bijection from $\mathbb{H}\setminus K_t$ onto $\mathbb{H}$. By definition for any $z \in \mathbb{H}\setminus K_t$, the solutions exist for $[0,t]$, hence g_t is well-defined on $\mathbb{H}\setminus K_t$. The time-reversed differential equation is $\dot{y}_t = \frac{1}{U_t - y_t}$. For this, $\operatorname{Im} \dot{y}_t = \frac{\operatorname{Im} y_t}{|U_t - y_t|^2}$ which shows that if $y_0 \in \mathbb{H}$, the the solution exists for all time and $y_t \in \mathbb{H}$. This provides the inverse of g_t on $\mathbb{H}$.

Next we show that g_t is analytic. We just sketch the idea. Let $\dot{x}_t = F(x_t, t)$ be a differential equation with F holomorphic in the first variable and continuous in the second. Recall the method of Picard's iteration to get a solution (locally). There we start with $X_0(t,x) = x$ for $x \in \mathbb{C}$ and define

$$X_{n+1}(t,x) = x + \int_0^t F(X_n(s,x), s)\, ds.$$

All this will be done locally. Inductively it is clear that each X_n is analytic in x. Picard's iterates converge locally uniformly to the solution of the ODE. Therefore the solution is also holomorphic.

Thus we have shown that g_t are conformal maps from $\mathbb{H}\setminus K_t$ onto $\mathbb{H}$. For fixed t and large z, we can argue (How?) that $\dot{g}_t(z) = 2/g_t(z) + O(g_t(z)^{-2})$ and hence $g_t(z) = z + \frac{2t}{z} + O(z^{-2})$. Thus $K_\bullet$ is in its natural parameterization. ■

Now we are ready to define SLE!

Definition 1.12.2. Let B be a standard one-dimensional Brownian motion. For $\kappa \geq 0$, let K_t be the increasing family of hulls with driving function $U_t := \sqrt{\kappa} B_t$ as assured by Theorem 1.12.1. The family of random hulls $K_\bullet$ is called SLE(κ).

It is a non-trivial theorem of Rohde and Schramm that SLE(κ)is generated by a random curve as in Exercise 1.11.2. Usually that curve γ is called SLE(κ). For many results, it suffices to think of SLE(κ)as a family of increasing hulls, but we shall anyway assume the result of Rohde and Schramm without proof and simply talk of SLE(κ)curves. Sometimes, the end result, K_∞ (or γ) is what we care about. It is a hull that connects 0 to ∞ (if $\infty \notin \overline{K}_\infty$, then K_∞ would be bounded and hence $a_t = 2t$ would be bounded above!).