2. ISOPERIMETRIC INEQUALITY

Isoperimetric nequality is a well-known statement in the following form: *Among all bodies in in space (in plane) with a given volume (given area), the one with the least surface area (least perimeter) is the ball (the disk).*

Several things need to be made precise. The notion of volume in space or area in the plane are understood to mean Lebesgue measure on $\mathbb{R}^3$ or $\mathbb{R}^2$ or more generally on $\mathbb{R}^d$ (we denote it by $m_d(A)$ o). Then of course we restrict the notion of "bodies" to Borel sets (or Lebesgue measurable sets).

Still, in measure theory class we (probably!) did not study the notion of surface area of a Borel set in $\mathbb{R}^3$ or the perimeter of a Borel set in $\mathbb{R}^2$. We first need to fix this notion. And then state a precise theorem. First we state a (stronger form) of the isoperimetric inequality which completely avoids the notion of surface area or perimeter.

Theorem 1 (Isoperimetric inequality)**.** *Let A be Borel subsets of $\mathbb{R}^d$ and let B be a closed ball such that $m_d(A) = m_d(B)$. Then, for any $\varepsilon > 0$, we have $m_d(A_\varepsilon) \geq m_d(B_\varepsilon)$ where $A_\varepsilon = \{x \in \mathbb{R}^d : d(x,y) \leq \varepsilon$ for some $y \in A\}$.*

How does this relate to the informally stated version above? If at all we can define the surface area of A, it must be the limit (or lim sup or lim inf) of $(m_d(A_\varepsilon) - m_d(A))/\varepsilon$ as $\varepsilon \to 0$. For simplicity, let us define the surface area (or "perimeter") of a Borel set $A \subseteq \mathbb{R}^d$ as

$$peri(A) := \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon| - |A|}{\varepsilon}$$

which is either a non-negative real number or $+\infty$. Then, Theorem 1 clearly gives the following theorem as a corollary.

Theorem 2 (Isoperimetric inequality - standard form)**.** *Let A be Borel subsets of $\mathbb{R}^d$ and let B be a closed ball such that $m_d(A) = m_d(B)$. Then, $peri(A) \geq peri(B)$.*

In this sense, we are justified in saying that Theorem 1 is stronger than Theorem 2 which is what we had in mind when we started out. In addition, note the great advantage of the former being easy to state for all Borel sets without having to define the notion of surface area (which would restrict the class of sets further). However, we have omitted a key point in the isoperimetric inequality.

Theorem 3 (Equality in isoperimetric inequality)**.**

(1) In the setting of Theorem 1 assume that A is closed. If $|A_\varepsilon| = |B_\varepsilon|$ for some $\varepsilon > 0$, then $A = B(x,r)$ for some $x \in \mathbb{R}^d$.

(2) In the setting of Theorem 2 assume that A is closed. If $peri(A) = peri(B)$, then $A = B(x,r)$ for some $x \in \mathbb{R}^d$.

A point to note is that the first statement does not imply the second![2]

We present two proof of Theorem 1. A short one using the Brunn-Minkowski inequality and a longer but more natural one by Steiner symmetrization.

2.1. Brunn-Minkowski inequality and a first proof of isoperimetric inequality.

For simplicity write $|A|$ for $m_d(A)$. For $A, B \subseteq \mathbb{R}^d$, define their Minkowski sum $A + B := \{a + b : a \in A, b \in B\}$.

Theorem 4 (Brunn-Minkowski inequality)**.** *Let A, B be non-empty Borel subsets of $\mathbb{R}^d$. Then $|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}$.*

It is very easy to prove in one dimension. In fact, it is a continuous analogue of the following inequality that we leave as an exercise.

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[2] Let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be two smooth function with $f(0) = g(0) = 0$. Consider the two sets of statements:

(1) (A) $f'(0) \geq g'(0)$ and (A') $f(x) \geq g(x)$ for small enough x.

(2) (B) $f'(0) \geq g'(0)$ and (B') $f(x) \geq g(x)$ for small enough x.

While (A') is stronger than (A), note that (B) is stronger than (B')! Meditate on the analogy between this situation and the relative strengths of Theorem 1 and Theorem 2.

Exercise 5 (Cauchy-Davenport inequality)**.** Let A,B be non-empty finite subsets of $\mathbb{Z}$. Then $|A+B| \geq |A| + |B| - 1$ and the inequality cannot be improved (here $|A|$ denotes the cardinality of A).

Proof of Theorem 1 using Brunn-Minkowski inequality. Assume $|A| = |rB|$ where B is the unit ball and $r > 0$. Then $A_\varepsilon = A + \varepsilon B$ and hence by Brunn-Minkowski

$$\begin{aligned} |A_\varepsilon|^{1/d} &\geq |A|^{1/d} + \varepsilon |B|^{1/d} \\ &= r|B|^{1/d} + \varepsilon |B|^{1/d} \\ &= |(r+\varepsilon)B|^{1/d}. \end{aligned}$$

Since $(rB)_\varepsilon = (r+\varepsilon)B$, we have proved that $|A_\varepsilon| \geq |(rB)_\varepsilon|$ as required. ■

Proof of Brunn-Minkowski inequality. **Case 1:** Suppose $A = \mathbf{x} + [0,a_1] \times \ldots \times [0,a_d]$ and $B = \mathbf{y} + [0,b_1] \times \ldots \times [0,b_d]$ are any two parallelepipeds with sides parallel to the axes (we shall refer to them as standard parallelepipeds). Then $A + B = \mathbf{x} + \mathbf{y} + [0, a_1 + b_1] \times \ldots \times [0, a_d + b_d]$. Thus,

$$\begin{aligned} \frac{|A|^{1/d} + |B|^{1/d}}{|A+B|^{1/d}} &= \left(\prod_{k=1}^{d} \frac{a_k}{a_k + b_k} \right)^{1/d} + \left(\prod_{k=1}^{d} \frac{b_k}{a_k + b_k} \right)^{1/d} \\ &\leq \frac{1}{d} \sum_{k=1}^{d} \frac{a_k}{a_k + b_k} + \frac{1}{d} \sum_{k=1}^{d} \frac{b_k}{a_k + b_k} \qquad \text{(AM-GM inequality)} \\ &= 1. \end{aligned}$$

Case 2: Suppose $A = A_1 \sqcup \ldots \sqcup A_m$ and $B = B_1 \sqcup \ldots \sqcup B_n$ are finite unions of standard parallelepipeds with pairwise disjoint interiors. When $m = n = 1$ we have already proved the theorem. By induction on $m+n$, we shall prove it for all $m, n \geq 1$.

Translating A or B does not change any of the quantities in the inequality, hence we may freely do so. Suppose $m \geq 2$. We leave it as an exercise to check the following claim.

Claim: There is at least one axis direction $i \leq d$ and a number $t \in \mathbb{R}$ such that each of the sets $A' := A \cup \{x : x_i \leq t\}$ and $A'' := A \cap \{x : x_i < t\}$ is a union of atmost $m-1$ standard parallelepipeds with pairwise disjoint interiors.

Set $\lambda = |A'|/|A|$. By the above claim, $0 < \lambda < 1$ and each of A' and A'' is a disjoint union of at most $m-1$ parallelepipeds (with sides parallel to the axes). Now translate B along the ith direction, i.e., for each s consider $B_s := B + s\mathbf{e}_i$ and let $B'_s = B_s \cap \{x : x_i \leq t\}$ and $B''_s = B_s \cap \{x : x_i \geq t\}$. Choose the unique s (why?) such that $|B'_s|/|B| = \lambda$ and set $B' = B'_s$ and $B'' = B''_s$.

By the inductive hypothesis,

$$\begin{aligned} |A' + B'| &\geq \left(|A'|^{1/d} + |B'|^{1/d} \right)^d = \lambda \left(|A|^{1/d} + |B|^{1/d} \right)^d, \\ |A'' + B''| &\geq \left(|A''|^{1/d} + |B''|^{1/d} \right)^d = (1 - \lambda) \left(|A|^{1/d} + |B|^{1/d} \right)^d. \end{aligned}$$

Further, observe that $A' + B' \subseteq \{x : x_i \leq t\}$ and $A'' + B''\{x : x_i \geq t\}$ and hence $|(A' + B') \cap (A' + B')| = 0$. Therefore,

$$\begin{aligned} |A + B| &= |A' + B'| + |A'' + B''| \\ &= \lambda \left(|A|^{1/d} + |B|^{1/d} \right)^d + (1 - \lambda) \left(|A|^{1/d} + |B|^{1/d} \right)^d \\ &= \left(|A|^{1/d} + |B|^{1/d} \right)^d. \end{aligned}$$

This completes the proof when A, B are finite unions of standard parallelepipeds.

Case 3: Let A and B be compact sets. Let $Q = [-1,1]^d$ and fix $\varepsilon > 0$. Observe that compactness of A implies that there exist $x_1, \dots, x_n \in A$ (for some n) such that $A \subseteq \tilde{A}$ where $A'' = \cup_{i=1}^n (x_i + \varepsilon Q)$. It is easy to see that $A'' \subseteq A_{\varepsilon\sqrt{d}}$ and that A'' may be written as a finite union of standard rectangles whose interiors are pairwise disjoint. Similarly find $B'' = \cup_{j=1}^m (y_j + \varepsilon Q)$ that is a union of standard rectangles whose interiors are pairwise disjoint and such that $B \subseteq B'' \subseteq B_{\varepsilon\sqrt{d}}$.

Then, observe that $A'' + B'' \subseteq (A+B)_{2\sqrt{d}\varepsilon}$. Since A'' and B'' are finite unions of standard parallelepipeds, by the previous case, we know that Brunn-Minkowski inequality applies to them. Thus,

$$\begin{aligned} |(A+B)_{2\sqrt{d}\varepsilon}| &\geq |A'' + B''| \\ &\geq (|A''|^{1/d} + |B''|^{1/d})^d \\ &\geq (|A|^{1/d} + |B|^{1/d})^d. \end{aligned}$$

This is true for every $\varepsilon > 0$. As $A + B$ is compact (why?), we see that $|(A+B)_{2\sqrt{d}\varepsilon}| \downarrow |A+B|$ as $\varepsilon \downarrow 0$. Brunn-Minkowski inequality has thus been proved for all compact subsets A, B.

Case 4: Let A and B be general Borel sets. If either of A or B has infinite Lebesgue measure, there is nothing to prove. Otherwise, by regularity of Lebesgue measure, there are compact sets $A' \subseteq A$ and $B' \subseteq B$ such that $|A \setminus A'| < \varepsilon$ and $|B \setminus B'| < \varepsilon$. Then of course $A + B \supseteq A' + B'$ and hence

$$|A+B|^{1/d} \geq |A'+B'|^{1/d} \geq |A'|^{1/d} + |B'|^{1/d} \geq (|A| - \varepsilon)^{1/d} + (|B| - \varepsilon)^{1/d}.$$

Letting $\varepsilon \to 0$ we get the inequality for A and B. ■

2.2. **Equality and stability.** In any equality, one would like to know sharp conditions when equality is attained. The following results give such conditions for the isoperimetric inequality and Brunn-Minkowski inequality.

In isoperimetric inequality, the ball is essentially the unique minimizer. However, sets of measure zero may be removed (sometimes added) without changing the volume or perimeter, hence the statement is a little more nuanced.

Result 6. Let A be a non-empty Borel set in $\mathbb{R}^d$ such that $|A| = |B_r|$. If $peri(A) = peri(B_r)$, then there is some $x \in \mathbb{R}^d$ such that $|A \triangle B(x,r)| = 0$. In particular, if there is some $\varepsilon > 0$ such that $|A_\varepsilon| = |(B_r)_\varepsilon|$, then again there is an $x \in \mathbb{R}^d$ such that $|A \triangle B(x,r)| = 0$.

When is equality attained in the Brunn-Minkowski inequality. Here are some simple exercises to get an idea.

Exercise 7.
(1) Suppose A is a convex set. Show that with $B = A$, equality is attained in Brunn-Minkowski inequality.
(2) Show that even if $B = A$, equality is not attained in general.
(3) If K is a compact convex set and $A = x + \alpha K$ and $B = y + \beta K$ for some $x, y \in \mathbb{R}^d$ and $\alpha, \beta > 0$, show that equality is attained in Brunn-Minkowski.
(4) If A and B are standard rectangles, show that equality is attained if and only if A and B are homothetic to each other (i.e., they are as described in the previous part of the exercise).

Essentially, the third part of the exercise gives the only condition under which equality can hold. As before, there are zero measure sets to take care of, as given below.

Result 8. Let A, B be non-empty Borel subsets of $\mathbb{R}^d$ having finite Lebesgue measure. If $|A+B|^{1/d} = |A|^{1/d} + |B|^{1/d}$. Then, there exists a compact convex set K and positive numbers α, β and $x, y \in \mathbb{R}^d$ such that

$$|A \triangle (x + \alpha K)| = 0 \quad \text{and} \quad |B \triangle (y + \beta K)| = 0.$$

We have not proved these results. Later, we shall give a different proof of the isoperimetric inequality in which the equality case shall be resolved too. Instead we shall now introduce a stronger form of the statement of equality, known as stability. The underlying idea is very general and is very much the flavour of theorems in analysis today. The question is of this.

Question: Suppose A *nearly* achieves equality in the isoperimetric inequality. Is it true that A is *nearly* a ball?

Why does one care for such a statement? One way to explain would be to go back to the physical situation of raindrops. We mentioned that raindrops are spherical so as to minimize surface area. But all these are only nearly valid, since, by the atomic nature of matter, at the smallest scales, there is no sphere, no ball, no equality - all these are only approximate notions. As such, if there was a completely different shape that whose surface area is only $(1+10^{-50})$ times the surface area of a ball of the same volume, then would a raindrop know the difference? If not, should we not see drops of such shape also in nature? Stability theorems come to our rescue, and say that in such a case, the drop must be nearly a ball[3].

Suppose A is a Borel set in $\mathbb{R}^d$ with finite Lebesgue measure. Then, there is some r such that $|A| = |B_r|$. Define

$$\alpha(A) := \inf_{x\in\mathbb{R}^d} \frac{|A \triangle B(x,r)|}{|A|}, \quad \text{and} \quad \delta(A) := \frac{peri(A)}{peri(B_r)} - 1.$$

The idea is that $\delta(A)$ measures how far A is from achieving equality in isoperimetric inequality and $\alpha(A)$ measures how close A is to a ball.

Result 9. [Fusco, Maggi, Pretalli (2008)] Suppose A is a Borel set in $\mathbb{R}^d$ with finite Lebesgue measure. Then, $\alpha(A) \le C_d\sqrt{\delta(A)}$, where C_d is a constant that depends only on the dimension d.

One can also ask for such a stability theorem in Brunn-Minkowski inequality. I recommend the lecture by Alessio Figalli in ICM 2014 on this topic.

[3] Just to get the point across, consider the (special case of) Cauchy-Schwarz inequality. If $a_1,\dots,a_n$ are real numbers, then $\left(\frac{1}{n}\sum_{i=1}^n a_i\right)^2 \le \frac{1}{n}\sum_{i=1}^n a_i^2$ with equality if and only if a_i are all equal. Here is a stability result for this inequality. Define

$$\delta(a_1,\dots,a_n) = 1 - \frac{\left(\frac{1}{n}\sum_{i=1}^n a_i\right)^2}{\frac{1}{n}\sum_{i=1}^n a_i^2} \quad \text{and} \quad \alpha(a_1,\dots,a_n) = \sqrt{\frac{1}{n^2}\sum_{i,j=1}^n (a_i - a_j)^2}.$$

Here δ measures how close to equality is achieved in the Cauchy-Schwarz and α measure how close to equal are the a_is (in some averaged sense). Then, $\alpha(a_1,\dots,a_n) \le \sqrt{2\delta(a_1,\dots a_n)}$ (why?).