A *topological group* is a group G endowed with a Hausdorff topology such that the map $g \to g^{-1}$ (from G to G) and the map $(g, h) \to gh$ (from $G \times G$ to G) are continuous. Examples are $(\mathbb{R}, +)$, S^1, $U(n)$ (set of $n \times n$ unitary matrices), $SL_n(\mathbb{R})$ (the space of $n \times n$ matrices with determinant 1), the group of isometries of $\mathbb{R}^n$, any countable group (with discrete topology) etc.

With topology comes the Borel sigma-algebra (Hausdorff condition implies that singletons are measurable sets). A fundamental (in fact the starting point) of measure theory is the existence of Lebesgue measure, i.e., a (unique) translation-invariant regular measure.

Question: If G is a topological group, does there exist a non-trivial regular Borel measure μ on G that is invariant under left-translations. In symbols, we need a measure μ on the Borel sigma algebra such that $\mu(gA) = \mu(A)$ for all $g \in G$ and all $A \in \mathcal{B}_G$. How unique is it? Such a measure will be called (left) Haar measure.

Answer: If G is locally compact, there is such a measure and it is unique up to multiplication by positive constants. The left Haar measure may not be a right Haar measure.

Exercise 1. Assuming the answer stated above, show that when G is compact, any left Haar measure is also a right Haar measure.

Example 2. When $G = S^1$, the Lebesgue measure is the Haar measure. When G is countable, counting measure is the Haar measure. If $G = \mathbb{R} \setminus \{0\}$ under multiplication, then $dx/|x|$ is the Haar measure (check!).

In some specific cases we can construct Haar measures by hand as in the following exercise.

Exercise 3. Let $G = GL_n(\mathbb{R})$ be the group of $n \times n$ invertible matrices with real entries. It is an open set in $\mathbb{R}^{n^2}$ and hence it has positive Lebesgue measure (from $\mathbb{R}^{n^2}$). Let $dm(g)$ denote the restriction of the Lebesgue measure to $GL_n(\mathbb{R})$. Define a new measure μ by $d\mu(g) = |\det(g)|^{-n} dm(g)$. Show that μ is a Haar measure (first understand why $m(\cdot)$ is not!).

But proving that Haar measure exists for a general locally compact topological group is not straight forward. We shall prove it for compact groups.

Theorem 4 (von Neumann). *Let G be a compact group. Then a unique Haar measure exists (it is both left and right invariant).*

Henceforth we assume that G is compact.

Some preliminaries: Measure and integral are closely related. Recall

Result 5 (Riesz's representation theorem). Let X be a compact Hausdorff space and let $L : C(G) \to \mathbb{C}$ be a positive linear functional. Then there exists a unique finite Borel measure μ on X such that $Lf = \int f d\mu$ for all $f \in C(G)$.

The converse statement that if μ is a finite Borel measure, then $f \to \int f d\mu$ is a positive linear functional on $C(G)$ is straightforward, but to be noted.

Thus, instead of constructing a measure, we may construct a positive linear functional on $C(G)$. What does the invariance of Haar measure correspond to in terms of the corresponding linear functional? The following exercise answers this question. Let $\tau_h(g) = f(h^{-1}g)$.

Exercise 6. Let G be a compact group. Let the positive linear functional L and the Borel measure μ correspond to each other as in Riesz's representation theorem. The following are equivalent (assume that the group is compact, for simplicity).

(1) μ is a left-Haar measure on G, i.e., $\mu(gA) = \mu(A)$ for all $g \in G, A \in \mathcal{B}_G$.

(2) L is invariant, i.e., $L(\tau_h f) = L(f)$ for all $f \in C(G)$ and for all $h \in G$.

In summary, to prove Theorem 4, it suffices to construct a positive linear functional L on $C(G)$ such that $L(\tau_h f) = L(f)$.

Remark 7. In class, to isolate the main ideas and see them clearly, we first made the extra assumption that the topology on G is induced by an invariant metric d, i.e., $d(x,y) = d(gx,gy)$ for all $g,x,y \in G$. In that case, the steps are as follows.

(1) For any finite $A \subseteq G$, define $L_A f = \frac{1}{|A|} \sum_{a \in A} f(a)$. This is the positive linear functional corresponding to the atomic measure $\frac{1}{|A|} \sum_{a \in A} \delta_a$.

(2) Fix $\varepsilon > 0$ and show that if A and B are two ε-nets having minimal cardinality (i.e., $\cup_{a \in A} B(a, \varepsilon) = G$ and similarly for B etc.), then there is a bijection $\pi : A \to B$ such that $d(a, \pi(a)) \le 2\varepsilon$ for all $a \in A$.

(3) Deduce that if A (respectively B) is a minimal cardinality ε-net (respectively, δ-net), then $|L_A f - L_B f| \le \omega_f(\varepsilon) + \omega_f(\delta)$ where $\omega_f(\varepsilon) = \sup\{|f(x) - f(y)| : d(x,y) \le \varepsilon\}$.

(4) Deduce that for any choice of minimal cardinality ε-nets A_ε, the limit $\lim_{\varepsilon \to 0} L_{A_\varepsilon} f$ exists and is independent of the choice of the ε-nets.

(5) Define the limit above as Lf. Show that L is a positive linear functional on $C(G)$ and has the invariance property (left and right). The corresponding measure is a Haar measure.

The key step in the whole proof is the second step where Hall's marriage theorem is invoked to show the existence of a bijection with desired properties. In these notes, with this brief outline, we jump directly to the case of general compact groups (that may not be metrizable). The steps are analogous to the above, except that ε-nets do not make sense, and instead we work with all neighbourhoods of the identity.

4.1. **Some preliminaries before the proof of Theorem 4.** If V is an open neighbourhood of the identity e, then let $H_V := \{gVh : g, h \in G\}$, a collection of open subsets in G. Two elements $a, b \in G$ are said to be adjacent in H_V if there is some $X \in H_V$ that contains both a and b and then we write $a \sim_V b$. A set $A \subseteq G$ is called a V-blocking set[4] if $A \cap gVh \ne \emptyset$ for all $g, h \in G$. Given $f \in C(G)$, define $\omega_f(V) := \sup\{|f(g) - f(g')| : g, g' \text{ are adjacent } H_V\}$

Exercise 8. Use compactness of G and continuity of f to show that for any $\varepsilon > 0$, there exists V (a neighbourhood of e) such that $\omega_f(W) \le \varepsilon$ for any $W \subseteq V$.

For any finite multi-set $A = \{a_1, \ldots, a_n\}$ (multi-set means that a_i may be repeated), define the positive linear functional $L_A : C(G) \to \mathbb{R}$ by $L_A f := \frac{1}{n} \sum_{i=1}^{n} f(a_i)$ (this corresponds to the measure $\frac{1}{n} \sum_{i=1}^{n} \delta_{a_i}$).

The key idea is in the following lemma.

Lemma 9. *Fix V and let A and B be two blocking sets of H_V having the minimum possible cardinality (among all blocking sets). Then $|L_A f - L_B f| \le \omega_f(V)$ for all $f \in C(G)$.*

Proof. We can write $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. We claim that there is a bijection $\pi : A \to B$ such that $a \sim \pi(a)$ for each $a \in A$. Once we get such a π, we easily deduce that

$$|L_A f - L_B f| \le \sum_{a \in A} |f(a) - f(\pi(a))| \le \omega_f(V).$$

To produce the permutation π, we invoke Hall's marriage theorem! Its statement and proof are given in the next section for completeness.

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[4]General terminology: A hypergraph is a pair (V, E) where V is a set and E is a collection of non-empty subsets of V. Elements of V are called vertices and elements of E are called hyper-edges.

When each element of E has cardinality 2, the hypergraph is simply called a graph. A blocking set is a subset of V that intersects every hyper-edge.

Define a bipartite graph with vertex sets A and B (even if some element is common to both, they are regarded as distinct in this bipartite graph) by setting $a_i \sim b_j$ if a_i and b_j are adjacent in H_V. For a subset $A' \subseteq A$, let $N_{A'}$ be the set of all vertices in B adjacent to some vertex in A'. We claim that $C := (A \setminus A') \cup N_{A'}$ is a blocking set for H_V.

Indeed, consider any hyper-edge $S = gVh$. As A and B are blocking sets, there exists $a \in A \cap S$ and $b \in B \cap S$. If $a \in A \setminus A'$, then $a \in C$ and hence $C \cap S \neq \emptyset$. Otherwise, $a \in A'$ and b is anyway adjacent to a (since S contains both a and b). Consequently $b \in N_{A'}$ which again shows that $C \cap S \neq \emptyset$. Thus, C is blocking. ∎

Proof of Theorem 4. Consider $W \subseteq V$, neighbourhoods of e. Choose any blocking sets A and B for H_W and H_V, respectively. Then, we claim that $|L_A(f) - L_B(f)| \leq 2\omega_f(V)$.

To see this, let $C = \{ab : a \in A, b \in B\}$ and observe that $C = \sup_{a \in A} aB = \sup_{b \in B} Ab$. Clearly, Ab (respectively aB) is a blocking set (of minimal cardinality) for H_W (respectively H_V) for any b (respectively a). Hence, by Lemma 9 we deduce that

$$|L_A f - L_C f| \leq \frac{1}{|B|} \sum_{b \in B} |L_A f - L_{Ab} f| \leq \omega_f(W),$$

$$|L_B f - L_C f| \leq \frac{1}{|A|} \sum_{a \in A} |L_B f - L_{aB} f| \leq \omega_f(V).$$

Thus, $|L_A f - L_B f| \leq \omega_f(W) + \omega_f(V) \leq 2\omega_f(V)$ as claimed.

By an earlier exercise, give $\varepsilon > 0$, there is some V with $\omega_f(V) \leq \varepsilon$. Hence, any pair of numbers in the set $\{L_A f : A \text{ is a blocking set for } H_W \text{ for some } W \subseteq V\}$ are within 2ε of each other.

The collection of open neighbourhoods of e form a net[5]. The above considerations show that for any choice of blocking sets A_V of H_V, the limit $Lf := \lim_V L_A f$ exists and is independent of the choice of blocking sets. As a limit of positive linear functionals, L is also positive and linear. If A_V is a choice of blocking sets, so are $B_V := gA_Vh$ (fix any $g, h \in G$). But $L_{B_V} f = L_{A_V} f'$ where $f'(x) = f(gxh)$. Thus, $Lf = Lf'$ showing the invariance of the functional L. Hence, the corresponding measure is left and right invariant. ∎

5. MATCHING THEOREM

Theorem 10 (Hall's marriage theorem). *Let $G = (V, E)$ be a finite bipartite graph with parts V_1 and V_2 (so $V = V_1 \cup V_2$). For $A \subseteq V$, let $N(A)$ be the set of all vertices adjacent to some vertex of A. Then, the following are equivalent.*

(1) $|N(A)| \geq |A|$ for all $A \subseteq V_1$.

(2) There exists an injective mapping $f : V_1 \to V_2$ such that $f(x) \sim x$ for all $x \in V_1$.

In the case when $|V_1| = |V_2|$, clearly f must be a bijection (a complete matching).

We shall derive it from the following more general theorem on partially ordered sets (posets). Recall that a *chain* is a totally ordered subset of a poset and an *anti-chain* is a subset of which no two elements are comparable to each other.

Theorem 11 (Dilworth). *Let $(\mathcal{P}, \leq)$ be a finite partially ordered set. Then the following numbers are equal.*

(1) The minimal number of chains into which $\mathcal{P}$ can be decomposed (i.e., written as a union of).

(2) The maximal size of an anti-chain in $\mathcal{P}$.

You may look up the proof in many books, for example the excellent book *A Course in Combinatorics* by van Lint and Wilson.

Exercise 12. Derive Hall's theorem from Dilworth's theorem.

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[5]For two neighbourhoods V, W of e, say that $V \leq W$ if $V \supseteq W$ (note the reversing). Being a net means that this order is reflexive and transitive and that given V_1, V_2, there is an upper bound, namely $V_1 \cap V_2$ (also a neighbourhood of e). For a function on the net, for example, $V \to h(V)$, we say that $\lim h(V) = a$ if given any $\varepsilon > 0$ there is a V such that for all $W \geq V$ we have $|h(W) - a| \leq \varepsilon$. In the above proof, we show a Cauchy-like criterion and deduce that a limit exists.

Proof. It is clear that the second number is bounded by the first. We only need to prove the other way inequality.

The inequality is obvious if $|\mathcal{P}| = 1$. Assume that the theorem it true if $|\mathcal{P}| \leq n$.

Now suppose $\mathcal{P} = n$ and the maximal size of an anti-chain in $\mathcal{P}$ is m. Let C be a maximal (under inclusion) anti-chain of $\mathcal{P}$ and let $\mathcal{P}_1 = \mathcal{P} \setminus C$ so that $|\mathcal{P}_1| < |\mathcal{P}|$. If the maximal size of an anti-chain of $\mathcal{P}_1$ is m_1, then by the inductive hypothesis $\mathcal{P}_1$ can be decomposed into m_1 chains. Together with C, this gives a decomposition of $\mathcal{P}$ into $m_1 + 1$ chains.

Thus, if $m_1 \leq m - 1$, then we are done. Otherwise, $\mathcal{P}_1$ has an anti-chain $\{x_1, \ldots, x_m\}$. Let $\mathcal{P}_2 = \{x \in \mathcal{P} : x \geq x_i \text{ for some } i\}$. Since C is maximal and $x_i \notin C$ for all i, it follows that $C \nsubseteq \mathcal{P}_2$. In particular, $|\mathcal{P}_2| < n$ and the size of a maximal anti-chain in $\mathcal{P}_2$ is m (as $\{x_1, \ldots, x_m\}$ is an anti-chain in $\mathcal{P}_2$). By induction, write $\mathcal{P}_2$ as a union of m chains $C'_1, \ldots, C'_m$ such that x_i is the maximal element of C'_i. In exactly the same way, taking $\mathcal{P}_3 = \{x \in \mathcal{P} : x \leq x_i \text{ for some } i\}$, we decompose it into m chains $C''_1, \ldots, C''_m$ such that x_i is the minimal element of C''_i.

It is clear that $C'_i \cup C''_i$ is a chain for each i and their union is $\mathcal{P}_2 \cup \mathcal{P}_3 = \mathcal{P}$. This completes the proof. ∎

Proof of Hall's marriage theorem. Let $\mathcal{P} = V_1 \cup V_2$ with the partial order $a \leq b$ if $a \in V_1$, $b \in V_2$ and b is adjacent to a. This makes $\mathcal{P}$ a poset.

Consider any anti-chain of $\mathcal{P}$ and write it in the form $A_1 \cup A_2$ where $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$. By the anti-chain condition, $A_2 \cap N(A_1) = \emptyset$ and by the given conditions in the theorem, $|N(A_1)| \geq |A_1|$ and hence $|A_2| \leq$ ∎