SOME PRACTISE PROBLEMS

The problems in magenta could be hard. Don't worry if you cannot solve them.

Problem 1. Show that there is an irrational number α such that $\{\overline{\alpha^n} : n \geq 1\}$ (here $\overline{x}$ denotes $x \pmod 1$) is not equidistributed. In fact produce an α such that most of these numbers are close to 0.

Problem 2. Fix a norm on $\mathbb{R}^d$. Let B is a ball in this norm and A is a Borel set such that $|A| = |B|$ (Lebesgue measures are equal). Then $|A_\epsilon| \geq |B_\epsilon|$, where A_ϵ denotes the ϵ-neighbouhood of A in the given norm.

Problem 3. Deduce the Brunn-Minkowski inequality $|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}$, again for all compact A, B from the following statements (they are equivaelnt forms).

(1) $|A+B| \geq \min\{|A|, |B|\}$ for all compact A, B.

(2) If A_1, A_2 are compact sets, B_1, B_2 are balls and $|A_i| = |B_i|$ for $i = 1, 2$, then $|A_1 + A_2| \geq |B_1 + B_2|$.

Problem 4. Use Jensen's formula to deduce that a complex polynomial must have a zero in the complex plane.

Problem 5. Let K be a compact convex set of unit volume in $\mathbb{R}^d$. Then there is a point in K such that for any hyperplane ($n-1$ dimensional affine subspace) passing through x, the two parts $K_\pm$ of K on either side of the hyperplane have volume greater than or equal to $\left(\frac{d}{d+1}\right)^d$. [Eg. If K is an equilateral triangle and x is its centroid, then a line through the centroid parallel to one of the sides cuts the triangle into parts having areas $1/4$ and $3/4$.]

Problem 6. If A and B are compact convex sets in $\mathbb{R}^d$ and ℓ is any line, then $\sigma_\ell(A+B) \supseteq \sigma_\ell(A) + \sigma_\ell(B)$.

Problem 7. Let f be an entire function such that $|f(z)| \leq e^{|z|^p}$ for $|z| > R$ for some $R < \infty$. Let $\alpha_1, \alpha_2, \dots$ be the zeros of f (assume that 0 is not a root of f). Then show that $\sum_{n=1}^{\infty} \frac{1}{|\alpha_n|^q} < \infty$ for all $q > p$.

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