UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 1

13 August, 2012

1. Show that there exist infinitely many rational numbers between two distinct rational numbers.

2. Let A_i, $i = 1, 2, \ldots$, be countable sets. Let $A = \cup_{i=1}^{\infty} A_i$. Show that A is countable. Deduce that $\mathbb{Q}$,the set of rational numbers, is countable.

3. Prove that every infinite set contains a countable set.

4. Give an example of a countable bounded subset A of $\mathbb{R}$ whose supremum and infimun are both in A^c.

5. If A is a nonempty bounded subset of $\mathbb{R}$, and the infimum for A is equal to the supremum for A, what can you say about A?

6. Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences. Show that:

   (a) $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$;

   (b) $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$;

   (c) $\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$.

   (d) Suppose further that $\lim_{n\to\infty} y_n \neq 0$. Then show that
   $$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

   (e) Let c be a real number. Show that
   $$\lim_{n\to\infty} (cx_n) = c \lim_{n\to\infty} x_n.$$

7. Find $N \in \mathbb{N}$ such that
   $$|\frac{2n}{n+3} - 2| < \frac{1}{5}, \quad n \geq N.$$

8. For each of the following sequences, prove that either the sequence has a limit or that the sequence does not have a limit.

   (a) $\{\frac{n^2}{n+5}\}$.

   (b) $\{\frac{3n}{n+7\sqrt{n}}\}$.

   (c) $\{\frac{3n}{n+7n^2}\}$.

9. Show that if a sequence is convergent, then its limit is unique.

10. Let $\{x_n\}$ be a sequence of real numbers and

$$\lim_{m\to\infty} x_{2m} = l, \qquad \lim_{m\to\infty} x_{2m-1} = l$$

for some real number l. Prove that $x_n \to l$ as $n \to \infty$.