Solutions to Exercises 2

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1. (a) $|x + y| < 1$, i.e. $-1 < x + y < 1$.
   Thus it is the region in between the straight lines $x + y = 1$ and $x + y = -1$.

   (b) $x + 2y$ is integer, i.e. $x + 2y = n,\ n \in \mathbb{Z}$.
   They are the straight lines in $\mathbb{R}^2$ given by $x+2y = n$ where $n \in \mathbb{Z}$.

   (c) $1/(x + y)$ is non zero integer, i.e. $x + y = 1/n$ where $n \in \mathbb{Z} \setminus \{0\}$.
   So the points are the straight lines $x + y = 1/n$ where n is non zero integer.

   (d) $x^2 - 2xy + y^2 = 9$, i.e. $(x - y)^2 = 9$. So the set is the pair of straight lines given by $x - y = 3$ and $x - y = -3$.

   (e) $|x - 1| = |y - 1|$, i.e. $x - 1 = y - 1$ or $x - 1 = -y + 1$. So the points are the straight lines $x = y$ and $x + y = 2$.

   (f) $x = \sin y$, this the sine curve along y-axis.

   (g) $[x]$ it is the step function, the set say S can be described as $S = \cup_{n\in\mathbb{Z}}\{(x, n - 1) : n - 1 \le x < n\}$.

   (h) $\sqrt{x - [x]}$. For any x there exist $n \in \mathbb{Z}$ such that $n \le x < n + 1$.
   So $[x] = n$. Thus the set S of points can be described as
   $S = \cup_{n\in\mathbb{Z}}\{(x, \sqrt{x - [n]}) : n \le x < n + 1\}$.

   (i) $[1/x]$. For $x > 0$ and if $x > 1$, then $[1/x] = 0$, else there exists $n \in \mathbb{N}$ such that $1/n < x \le 1/(n - 1)$,so $[1/x] = n - 1$.
   Similarly if $x < 0$ and if $x \le -1$,then $[1/x] = -1$, else there exists $n \in \mathbb{N}$ such that $-1/(n - 1) < x \le -1/n$, so $[1/x] = -n$.

   (j) $\{x\}$. For any x there exists $n \in \mathbb{Z}$ such that $n-1/2 \le x < n+1/2$ then $\{x\} = n$. So the set S of points is described as
   $S = \cup_{n\in\mathbb{Z}}\{(x, n) : n - 1/2 \le x < n + 1/2\}$

   (k) $\{x\} + \{2x\}/2$. For any x there exists $n \in \mathbb{Z}$ such that $n - 1/2 \le x < n + 1/2$ then $\{x\} = n$, and $2n - 1 \le 2x < 2n + 1$, so

      i. $\{2x\} = 2n-1$ when $2n-1 \le 2x < 2n-1/2$ i.e $\{x\}+\{2x\}/2 = 2n - 1/2$

      ii. $\{2x\} = 2n$ when $2n-1/2 \le 2x < 2n+1/2$ i.e $\{x\}+\{2x\}/2 = 2n$

      iii. $\{2x\} = 2n + 1$ when $2n + 1/2 \le 2x < 2n + 1$ i.e. $\{x\} + \{2x\}/2 = 2n + 1/2$

   Thus the set S of points can be described as

$$\begin{aligned} S = &\cup_{n\in\mathbb{Z}} \{(x, 2n - 1/2) : n - 1/2 \le x < n - 1/4\} \\ &\cup_{n\in\mathbb{Z}} \{(x, 2n) : n - 1/4 \le x < n + 1/4\} \\ &\cup_{n\in\mathbb{Z}} \{(x, 2n + 1/2) : n + 1/4 \le x < n + 1/2\}. \end{aligned}$$

2. Discussed in class.

3. Given $|x - x_0| < \epsilon/2$ and $|y - y_0| < \epsilon/2$. Then

$$\begin{aligned} |(x+y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &< |x - x_0| + |y - y_0| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

and

$$\begin{aligned} |(x-y) - (x_0 - y_0)| &= |(x - x_0) - (y - y_0)| \\ &< |x - x_0| + |y_0 - y| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

4. Given $|x - x_0| < \min(\frac{\epsilon}{2(|y_0|+1)}, 1)$ and $|y - y_0| < \frac{\epsilon}{2(|x_0|+1)}$.
Since

$$\begin{aligned} |xy - x_0y_0| &= |xy - x_0y + x_0y - x_0y_0| \\ &\le |y||x - x_0| + |x_0||y - y_0| \\ &\le |y - y_0||x - x_0| + |y_0||x - x_0| + |x_0||y - y_0| \\ &< |y - y_0|(1 + |x_0|) + |y_0||x - x_0| \qquad \text{(Since } |x - x_0| < 1) \\ &< \frac{\epsilon}{2} + \frac{\epsilon|y_0|}{2(|y_0|+1)} < \epsilon. \end{aligned}$$

5. Given $y_0 \neq 0$ and $|y - y_0| < \min(|y_0|/2, \epsilon|y_0|^2/2)$.
Since

$$|y - y_0| < |y_0|/2 \Rightarrow -|y - y_0| > -|y_0|/2$$

and

$$|y_0| = |y_0 - y + y| \le |y - y_0| + |y_0|$$

so

$$\begin{aligned} |y| &\ge |y_0| - |y - y_0| \\ &> |y_0| - |y_0|/2 = |y_0|/2. \end{aligned}$$

Hence $y \neq 0$, and $\frac{1}{|y|} < \frac{2}{|y_0|}$. Now

$$|\frac{1}{y} - \frac{1}{y_0}| = \frac{|y - y_0|}{|y||y_0|} < \frac{2}{|y_0|}\frac{1}{|y_0|}(|y - y_0|) < \epsilon.$$

6. Consider $z = \frac{1}{y}$ and $z_0 = \frac{1}{y_0}$, then $|x/y - x_0/y_0| = |xz - x_0z_0|$.
From Exercise 4, it follows that if $|x - x_0| < \min(\frac{\epsilon}{2(|z_0|+1)}, 1)$ and $|z - z_0| < \frac{\epsilon}{2(|x_0|+1)}$, then $|xz - x_0z_0| < \epsilon$.

   Let $\epsilon_0 = \frac{\epsilon}{2(|x_0|+1)}$, then from Exercise 5, if $|y - y_0| < \min(|y_0|/2, \epsilon_0|y_0|^2/2)$, we have $|z - z_0| = |1/y - 1/y_0| < \epsilon_0$.

   Thus substituting the values of ϵ_0 and z_0 in terms of ϵ and y_0, we get the required condition as, if

$$|x - x_0| < \min(\frac{\epsilon|y_0|}{2(|y_0|+1)}, 1), |y - y_0| < \min(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{4(|x_0|+1)}).$$

   Then $|x/y - x_0/y_0| < \epsilon$.

7. (a) $\lim_{x\to 0^+} |x|/x$.
For given some $\epsilon > 0$ choose any $\delta > 0$, such that $0 < x < \delta$. Then $|x|/x = 1$, i.e $|(|x|/x) - 1| = 0 < \epsilon$.
Hence $\lim_{x\to 0^+} |x|/x = 1$.

   (b) $\lim_{x\to 0^-} [1/x]$.
Limit does not exists. To prove this we need to show that $\lim_{x\to 0^-} [1/x] \neq l$, for any $l \in \mathbb{R}$. i.e there exists an $\epsilon > 0$ such that for every $\delta > 0$ there is some x such that $|x| < \delta$ but $|[1/x] - l| > \epsilon$.
So let $l \in \mathbb{R}$, then there exists $n_0 \in \mathbb{N}$ such that $-n_0 < l+1$. Now for any $\delta > 0$ there exists $n_1 \in \mathbb{N}$ such that $1/n_1 < \delta$. Now choose $n \in \mathbb{N}$ such that $n > \max\{n_0, n_1\}$. Then let $x = -1/n$. So we have $|x| < \delta$, but $[1/x] = -n < -n_0 < l + 1$ i.e. $|[1/x] - l| > 1$.

   (c) $\lim_{x\to 0}(\sin x - \sin a)/(x - a)$

$$\begin{aligned}\frac{(\sin x - \sin a)}{(x-a)} &= \frac{\sin(x-a)\cos a + \cos(x-a)\sin a - \sin a}{(x-a)} \\ &= \frac{\cos a \sin(x-a)}{(x-a)} - \frac{\sin a(1-\cos(x-a))}{(x-a)^2}(x-a)\end{aligned}$$

   Then by part (d) for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\frac{\sin a(1-\cos(x-a))}{(x-a)^2} - \frac{\sin a}{2}| < \min\{\epsilon/6, 1\}$ whenever
$|x - a| < \delta$. Now if $|x - a| < \min\{\epsilon/4, \delta\}$ then by Exercise 4, $|\frac{\sin a(1-\cos(x-a))}{(x-a)^2}(x - a) - 0| < \epsilon/2$. Also for $\epsilon > 0$ there exist $\delta_1 > 0$ such that $|\frac{\sin(x-a)}{(x-a)} - 1| < \epsilon/2$ whenever $|x - a| < \delta_1$. Now $|\cos a| \leq 1$, so

$$|\frac{\cos a \sin(x-a)}{(x-a)} - \frac{\sin a(1-\cos(x-a))}{(x-a)^2}(x-a) - \cos a| < \epsilon$$
$$\text{whenever } |x - a| < \min\{\epsilon/4, \delta, \delta_1\}$$

Hence $\lim_{x\to 0}(\sin x - \sin a)/(x-a) = \cos a$.

(d) $\lim_{x\to 0}(1-\cos x)/x^2$.

$$\frac{(1-\cos x)}{x^2} = 2\frac{\sin^2(x/2)}{x^2} = \frac{1}{2}(\frac{\sin(x/2)}{x/2})^2$$

Now given $\lim_{x\to 0}\sin x/x = 1$ implies that for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|(\frac{\sin(x/2)}{x/2}) - 1| < \min(\frac{\epsilon}{2}, 1), \text{ whenever } |x/2| < \delta/2$$

Then by Exercise 4 ,

$$|\{\frac{\sin(x/2)}{x/2}\}^2 - 1| < 2\epsilon \text{ whenever } |x| < \delta$$
$$i.e. \ |\frac{(1-\cos x)}{x^2} - 1/2| < \epsilon \text{ whenever } |x| < \delta.$$

Hence $\lim_{x\to 0}(1-\cos x)/x^2 = 1/2$.

8. Given $\lim_{x\to 0} g(x) = 0$, i.e for ever $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x)| < \epsilon$ whenever $|x| < \delta$. Since $|\sin(1/x)| \le 1$ for any x, $|\sin(1/x)g(x)| \le |\sin(1/x)||g(x)|| < \epsilon$ whenever $|x| < \delta$.
Hence $\lim_{x\to 0}\sin(1/x)g(x) = 0$.

9. Given $f(x) \le g(x)$ for all x. Let $\lim_{x\to a} f(x) = \alpha_1$ and $\lim_{x\to a} g(x) = \alpha_2$. We have to show that $\alpha_1 \le \alpha_2$. Suppose not, i.e $\alpha_1 > \alpha_2$ then there exists $\delta > 0$ such that $|f(x)-\alpha_1| < (\alpha_1-\alpha_2)/2$ and $|g(x)-\alpha_2| < (\alpha_1-\alpha_2)/2$ whenever $|x-a| < \delta$. Solving in terms of α_1 and α_2

$$\frac{(\alpha_1+\alpha_2)}{2} < f(x) < \frac{(3\alpha_1-\alpha_2)}{2}, \quad \frac{(3\alpha_2-\alpha_1)}{2} < g(x) < \frac{(\alpha_1+\alpha_2)}{2}$$

So we get $g(x) < f(x)$ whenever $|x-a| < \delta$, which is a contradiction! Hence $\alpha_1 \le \alpha_2$.
Let $g(x) = f(x)+|x|$ on $\mathbb{R}\setminus\{0\}$ and $g(0) = f(0)+1$. Then $f(x) < g(x)$ but $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$. Thus it is not true that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

10. Given $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$, then for some $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x)-l| < \epsilon$ and $|h(x)-l| < \epsilon$ whenever $|x-a| < \delta$. So

$$l - \epsilon < f(x) \le g(x) \le h(x) < l + \epsilon,$$

Hence $|g(x) - l| < \epsilon$ whenever $|x-a| < \delta$ i.e $\lim_{x\to a} g(x) = l$.