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NOTE
A Note on Controllability of Impulsive Systems

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1. INTRODUCTION

In this short article we investigate complete controllability of the control system with impulse effects

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t) + B(t)u(t) + f(t, x(t)), \qquad t \neq t_k, t \in [t_0, T] \\ x(t_k^+) &= \left[I + D^k u(t_k)\right] x(t_k) \\ x(t_0) &= x_0, \end{aligned} \tag{1.1}$$

where, for each $t \in [t_0, T]$, the state $x(t)$ is an n-vector, control $u(t)$ is an m-vector, $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, respectively, with piecewise continuous entries, and $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_\rho < T$

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are the time points at which we give impulsive controls $u(t_k)$ to the system. For each $k = 1, 2, \ldots, \rho$, $D^k u(t_k)$ is an $n \times n$ diagonal matrix such that $D^k u(t_k) = \sum_{i=1}^{m} d_i^k u_i(t_k) I$, where I is the identity matrix on $\mathbb{R}^n$, and $d_i^k \in \mathbb{R}$, and f: $[t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function which is measurable with respect to the first argument and continuous with respect to the second argument. The control $u(t)$ is said to be impulsive if at $t = t_k$ the pulses are regulated and are chosen arbitrarily in the rest of the domain. Study of such a system received much attention in recent years due to the fact that many evolutionary processes experience an abrupt change of state at certain moments (refer to Lakshmikantham *et al.* [5]). In [6], Leela *et al.* studied the controllability property of a time-invariant unperturbed system (i.e., with $A(t) = A$, $B(t) = B$, and $f \equiv 0$ in (1.1)). In [6], it is stated that the time-invariant unperturbed system is always completely controllable. However, only controllability to the origin (null controllability) is established, and that in a rather tedious manner. In fact, controllability to the origin follows very easily if one notes that, at any arbitrary time point t_k, an impulsive control may be applied to the state $x(t_k^+)$, keeping other controls to be zero, so that the system is instantaneously driven to the origin.

We obtain conditions for complete controllability of unperturbed (i.e., with $f \equiv 0$) and perturbed systems separately. Section 2 deals with complete controllability of the unperturbed system, and in Section 3, some sufficient conditions for complete controllability of the perturbed system (1.1) are obtained.

2. CONTROLLABILITY OF THE UNPERTURBED SYSTEM

To study complete controllability of (1.1) we first study complete controllability of the corresponding unperturbed system

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t) + B(t)u(t), \qquad t \neq t_k, t \in [t_0, T] \\ x(t_k^+) &= \left[I + D^k u(t_k)\right] x(t_k) \\ x(t_0) &= x_0. \end{aligned} \tag{2.1}$$

In this section we prove the necessary and sufficient condition for the complete controllability of (2.1). As remarked earlier, Leela *et al.* [6] reported that the impulsive system (2.1) (with $A(t)$ and $B(t)$ time-invariant matrices) is always completely controllable, which is an incorrect assertion as is evident from the following example.

COUNTEREXAMPLE. Consider a two-dimensional system with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \rho = 1, t_0 = 0.$$

Let

$$D^1 u(t_1) = \begin{bmatrix} d_1^1 u(t_1) & 0 \\ 0 & d_1^1 u(t_1) \end{bmatrix}.$$

That is, the system has the form

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \qquad t \neq t_1, t \in [0, T]$$

$$\begin{pmatrix} x_1(t_1^+) \\ x_2(t_1^+) \end{pmatrix} = \begin{bmatrix} 1 + d_1^1 u(t_1) & 0 \\ 0 & 1 + d_1^1 u(t_1) \end{bmatrix} \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \end{pmatrix}.$$

Let $(0,1)^T$ be the initial state and let $(1,0)^T$ be the desired final state. With this initial state the state at time $t = T$ is given by

$$\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + d_1^1 u(t_1) \end{pmatrix} + \int_0^{t_1} \begin{pmatrix} 0 \\ 1 + d_1^1 u(t_1) \end{pmatrix} u(s)\, ds$$
$$+ \int_{t_1}^{T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(s)\, ds.$$

Clearly, no control $u(t)$ will steer $(0,1)^T$ to $(1,0)^T$. Therefore this system is not completely controllable.

What exactly is proved in [6] is the null controllability (i.e., controllable to the origin from any initial state). We note that the null controllability of (2.1) can be proved in a few lines. For, if we prescribe $u(t)$ at $t = t_k$ such that

$$\prod_{k=j}^{\rho} \left(I + D^k u(t_k)\right) = 0 \qquad \text{for } 1 \leq j \leq \rho$$

$$u(t) = 0 \qquad \text{for all } t \neq t_k$$

then the solution of (2.1) given by

$$x(t) = \phi(t, t_0) \prod_{t_0 < t_k < t_\rho} \left[I + D^k u(t_k) \right] x_0$$

$$+ \sum_{i=1}^{\rho} \int_{t_{i-1}}^{t_i} \phi(t, s) \prod_{t_{i-1} < t_k < t} \left(I + D^k u(t_k) \right) B(s) u(s)\, ds$$

$$+ \int_{t_\rho}^{t} \phi(t, s) B(s) u(s)\, ds \tag{2.2}$$

satisfies $x(T) = 0$.

That is, the system (2.1) is always null controllable without any conditions on $A(t)$ and $B(t)$. It can be shown that something more is true for the impulsive system. Any initial condition $x_0 \in \mathbb{R}^n$ can be steered to any desired state x_1, if

$$x_1 \in \text{Range}(C) + \text{Span}\left(\phi(T, t_0) x_0 \right), \tag{2.3}$$

where C: $L^2(I, \mathbb{R}^m) \to \mathbb{R}^n$ is the linear operator defined by

$$Cu = \int_{t_\rho}^{T} \phi(T, s) B(s) u(s)\, ds. \tag{2.4}$$

Obviously, $0 \in \text{Range}(C) + \text{Span}(\phi(T, t_0) x_0)$ for arbitrary $x_0 \in \mathbb{R}^n$, and this justifies the null controllability of (2.1). We now give the following characterization for the complete controllability of (2.1).

THEOREM 2.1. *The system* (2.1) *is completely controllable if and only if the controllability* Grammian $W(t_\rho, T)$ *defined by*

$$W(t_\rho, T) = \int_{t_\rho}^{T} \phi(T, \tau) B(\tau) B^*(\tau) \phi^*(T, \tau)\, d\tau \tag{2.5}$$

is non-singular.

Proof. For any initial state x_0 the solution $x(t)$ of (2.1) is given by (2.2). Since $\prod_{t_0 < t_k < t_\rho} (I + D^k u(t_k))$ is a diagonal matrix, it follows that

$$x(T) \in \text{Range}(C) + \text{Span}\left(\phi(T, t_0) x_0 \right).$$

The system (2.1) is completely controllable if and only if for every $x_0 \in \mathbb{R}^n$,

$$\text{Range}(C) + \text{Span}\left(\phi(T, t_0) x_0 \right) = \mathbb{R}^n.$$

This holds if and only if $\text{Range}(C) = \mathbb{R}^n$. Now the theorem follows directly from the fact that $\text{Range}(C) = \text{Range}\, W(t_\rho, T)$. Refer to Brockett [1].

For time-invariant system (2.1) we have the following Kalman rank condition to check complete controllability, which follows as a corollary of the above theorem.

COROLLARY 2.1. *Suppose that $A(t)$ and $B(t)$ are time-invariant matrices. Then* (2.1) *is completely controllable if and only if*

$$\mathrm{Rank}[B : AB : A^2B : \cdots A^{n-1}B] = n.$$

3. CONTROLLABILITY OF THE PERTURBED SYSTEM

We now give sufficient conditions for the complete controllability of the perturbed system (1.1). The solution of the system in the interval $[t_\rho, T]$ satisfies

$$x(t) = \phi(t, t_\rho)\tilde{x}_0 + \int_{t_\rho}^{t} \phi(t,\tau)B(\tau)u(\tau)\,d\tau + \int_{t_\rho}^{t} \phi(t,\tau)f(\tau, x(\tau))\,d\tau, \tag{3.1}$$

where $\tilde{x}_0$ is given by

$$\begin{aligned}\tilde{x}_0 = {} & \phi(t, t_0) \prod_{t_0 < t_k < t_\rho} [I + D^k u(t_k)] x_0 \\ & + \sum_{i=1}^{\rho} \int_{t_{i-1}}^{t_i} \phi(t,\tau) \prod_{t_{i-1} < t_k < t} [I + D^k u(t_k)] B(s)u(s)\,ds \\ & + \sum_{i=1}^{\rho} \int_{t_{i-1}}^{t_i} \phi(t,\tau) \prod_{t_{i-1} < t_k < t} [I + D^k u(t_k)] f(\tau, x(\tau))\,d\tau. \end{aligned} \tag{3.2}$$

Since we are looking for some sufficient conditions for complete controllability, let us first choose $u(t_k)$, $k = 1, 2, \ldots, \rho$, such that $[I + D^k u(t_k)] = 0$. Then (3.1) becomes

$$x(t) = \int_{t_\rho}^{t} \phi(t,\tau)B(\tau)u(\tau)\,d\tau + \int_{t_\rho}^{t} \phi(t,\tau)f(\tau, x(\tau))\,d\tau. \tag{3.3}$$

We assume throughout this section that f satisfies a growth condition

$$\|f(t,x)\| \le a\|x\| + b, \qquad \forall x \in \mathbb{R}^n,\ b > a \ge 0. \tag{3.4}$$

There are various sufficient conditions on f to guarantee that the nonlinear Volterra integral equation (3.3) has a unique solution for every fixed u. In this case we can define the solution operator

$$S\colon L^2(t_0, T; \mathbb{R}^m) \to L^2(t_0, T; \mathbb{R}^n)$$

by $Su = x$, where x satisfies (3.3) for a given u. The following lemma follows from Joshi and George [4] and George [2].

LEMMA 3.1. *Under each of the following cases the solution operator S is well defined and continuous.*

(a) *There exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|, \qquad \forall x, y \in \mathbb{R}^n. \tag{3.5}$$

(b) *f satisfies a growth condition* (3.4) *and there exists a constant $M(r) > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le M(r)\|x - y\|,$$
$$\forall x, y \in \mathbb{R}^n \text{ satisfying } \|x\|, \|y\| \le r. \tag{3.6}$$

(c) *f satisfies a growth condition* (3.4) *and there exists a constant $\beta > 0$ such that*

$$\langle f(t,x) - f(t,y), x - y\rangle \ge \beta\|x - y\|^2, \qquad \forall x, y \in \mathbb{R}^n \tag{3.7}$$

and

$$\|A(\cdot)\| > \beta.$$

Further, in case (a) *S is Lipschitz continuous, i.e., there exists a constant $\alpha > 0$ such that*

$$\|Su - Sv\| \le \alpha\|u - v\| \qquad \forall u, v \in L^2(t_0, T; \mathbb{R}^m). \tag{3.8}$$

In cases (b) *and* (c), *S satisfies a growth condition; that is, there exist constants $S_0, S \ge 0$ such that*

$$\|Su\| \le S\|u\| + S_0 \qquad \forall u \in L^2(t_0, T; \mathbb{R}^m). \tag{3.9}$$

Proof. See [2] and [4] for the proof.

Henceforth we assume that the solution operator S is well defined and satisfies either (3.8) or (3.9). Under this condition we obtain the following result for the complete controllability of (1.1).

THEOREM 3.1. *Suppose that*

(i) *$W(t_\rho, T)$ is nonsingular,*

(ii) *f is Lipschitz continuous* (*i.e., f satisfies* (3.5)),

(iii) *T and t_ρ are sufficiently close.*

Then the system (1.1) *is completely controllable.*

Proof. By (ii) S is well defined and satisfies (3.8). Now the complete controllability follows from the solvability of the equation

$$x_1 = \int_{t_\rho}^{T} \phi(T,\tau)B(\tau)u(\tau)\,d\tau + \int_{t_\rho}^{T} \phi(T,\tau)f(\tau,(Su)(\tau))\,d\tau. \quad (3.10)$$

Replacing u by $C^* v = B^*(t)\phi^*(T,t)W^{-1}(t_\rho, T)v$ in (3.10) we get

$$v = x_1 + Nv, \quad (3.11)$$

where $N\colon \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear operator defined by

$$Nv = -\int_{t_\rho}^{T} \phi(T,\tau)f(\tau,(SC^* v)(\tau))\,d\tau. \quad (3.12)$$

Therefore it suffices to prove that (3.11) has a solution for any $x_1 \in \mathbb{R}^n$. By (ii) and from Lemma 3.1, it can be shown that N is Lipschitz continuous and (iii) implies that N is a contraction. Therefore by the Banach contraction principle (3.11) has a unique solution. Hence the theorem follows.

When f is not uniformly Lipschitz continuous, we have the following theorem.

THEOREM 3.2. *Suppose that*

(i) *$W(t_\rho, T)$ is nonsingular,*

(ii) *f satisfies* (3.4),

(iii) *f satisfies either the monotonicity condition* (3.7) *or the local Lipschitz condition* (3.6),

(iv) *T and t_ρ are sufficiently close.*

Then the system (1.1) *is completely controllable.*

Proof. As in the case of Theorem 3.1, it suffices to show that (3.11) has a solution. By using Lemma 3.1 it is not difficult to show that N is a quasi-bounded operator. By (iv) it follows that the quasi-norm is strictly less than 1. Compactness of N can also be proved easily. Therefore, by Grana's theorem [3], (3.11) has a solution. Hence the system is completely controllable.

Remark 3.1. When the Lipschitz constant L in (3.5) or the growth constant a in (3.4) of f is sufficiently small, then the condition on the closeness of T and t_ρ can be removed in Theorems 3.1 and 3.2. Also, if f is uniformly bounded (i.e., there exists constant $M > 0$ such that $\|f(t,x)\| \leq M$) then the conditions (iii) of Theorem 3.1 and (iv) of Theorem 3.2 can be removed.

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