The Meaning of Integration – I

A K Nandakumaran

1. Introduction

Integration is the 'inverse of differentiation' is what we are told in our beginning calculus course. It is indeed the case when the given function is continuous[1]. Augustin Cauchy[2] (1789 - 1857), a famous mathematician, was the first person to establish that any continuous function is the derivative of a differentiable function. The proof of this result, known as the *Fundamental theorem of Calculus* (see below) is based on the concept of the area of planar regions bounded by 'suitable' curves. But in practice, it takes a few years of study to understand the concept of integration. In fact, one grasps the idea of integration only on studying the theory of Riemann[3] integration (R-integration) which is the generalization of Cauchy integration to the class of functions which are not necessarily continuous (*Figure* 1b).

A K Nandakumaran is an Associate Professor at the Department of Mathematics, Indian Institute of Science, Bangalore. Earlier, he was in TIFR Center, IISc Campus, Bangalore. His research interests include partial differential equations, homogenization and control problems.

As the concept of area lies at the very basis of integration, one should understand this concept in detail. Areas of planar regions or the volumes of solid geometrical objects are ancient concepts and even a layman has some ideas about it. Our goal in this article is to understand it mathematically without getting into complicated technicalities.

[1] Those who are not familiar with the technical term 'continuity' can think that the graph of such a function represents a 'nice' curve; that is an unbroken curve (see *Figure* 1a).

2. Real Number System

Although real numbers can be visualized as points on a line, mathematically a real number (even a natural number) is an abstract object. "God created the natural numbers; everything else is man's work". In these words, Leopold Kronecker (1823-1891) pointed out the safe ground on which the structure of mathematics can be built (see [2]). The construction of real numbers in-

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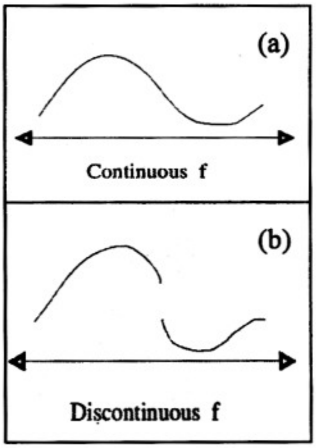


Figure 1.

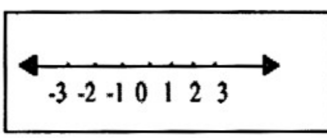
volves one of the most powerful and beautiful concepts of modern mathematics, namely the notion of 'completeness'. It is a very general notion and is useful in many situations including the development of areas and integration.

As remarked earlier, area is the basis of integration theory; just as counting is the basis of the real number system. We identify real numbers as points on a line. Consider a line on a plane and mark a point on the line by the symbol '0' and identify it as the origin. Then take a point at a unit distance of our convenience to the right of the origin, which we call the positive direction and denote it by the number '1' (*Figure* 2).

[2] Augustin-Louis Cauchy, the first of the great French mathematicians of the modern age, was born on August 21, 1789. Modern mathematics is indebted to Cauchy for two important reasons. The first was the introduction of rigor into mathematical analysis and the second was on the combinatorial theory. Seizing on the heart of Lagrange's method in the theory of equations, (see [1]) Cauchy made it abstract and began the systematic creation of the theory of groups.

Once this is done, from school level mathematics, we know that all integers (both positive and negative) and more generally, each number of the form $\frac{m}{n}, n \neq 0, m, n$ integers (known as rational numbers) corresponds to a unique point on the line. We call these points 'rational points'.

Observation: If P and Q, respectively represent two rational points given by $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$, $\frac{m_1}{n_1} < \frac{m_2}{n_2}$, then the mid point R of the line segment PQ represents another rational point given by $\frac{1}{2}(\frac{m_1}{n_1} + \frac{m_2}{n_2})$. In this fashion, one can construct infinitely many rational points (like the mid points of PR and RQ and so on) on the line segment PQ (see *Figure* 3). So the natural tendency is to conclude that every point on the line can be realized in this way. More precisely, every point on the line is a rational point, that is, it is at some rational distance from the origin, but it is not so. Soon, we realize that there are points on the line which do not correspond to rational points. The diagonal of a square of side 1, i.e.,



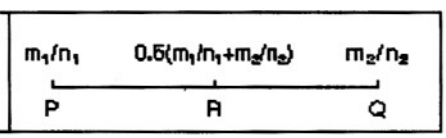


Figure 2(left).
Figure 3 (right).

$\sqrt{2}$ is a familiar example, from our high school, of such a number.

Warning : In fact, at this stage we do not know what $\sqrt{2}$ is? It can be shown[4] that there exists no rational number $\frac{m}{n}$ such that $(\frac{m}{n})^2 = 2$.

In other words, there are points on the line whose distance from the origin cannot be given by rational numbers. We call these points as 'gaps'. Thus $\sqrt{2}$ is a gap such that the square of its distance from the origin is 2.

The real number system is the realization of such 'gaps' in a rigorous mathematical sense. If the rational points on the line are considered along with the gaps, we will get the entire real line. Alternatively, the real numbers can be visualized as the 'completion' of the rational numbers. A very striking feature in the process of completion is that one can obtain a real number only by an 'infinite' process. These two terms are quite synonymous in the sense that, in general, the completion can be achieved only by an infinite process.

The fact is that even to obtain a point at a finite distance from the origin, one has to do an infinite process, thus making the 'real' numbers (both rational points and the gaps together) an abstract object. In other words, representing the gaps by the available quantities, namely, the rational numbers, involves an infinite process and hence a lot of mathematical technicalities.

There are various ways of visualizing the gaps. Two particular ways are 'Dedekind's Cut' and 'Cauchy Sequences'. The former is specific to the real number system, but the latter is a very general method. It is general in the sense that the concept of Cauchy sequences can be used to obtain completion of more abstract spaces. In the latter method, a gap is identified with a sequence of rational numbers. There are many sequences representing the same gap.

[3] George Friedreich Bernhard Riemann, the son of a Lutheran pastor was born in the little village of Breselenz, Germany, on September 17, 1826. This note is too small to write anything about him. What more evidence does one need than that the ideas of Riemannian Geometry conceived by him have found their physical interpretations in the theory of relativity developed by Albert Einstein. Since relativity, our view of physical sciences is not what it was before. Without the work of Riemann this revolutionary scientific thought would have been impossible; unless someone had created the concepts and the mathematical methods that Riemann created. As A S Eddington says (see [1]) "A geometer like Riemann might have foreseen the important features of the actual world". The well-known conjecture, better known as Riemann's Hypothesis, originated in his attempt to improve Legendre's empirical formula estimating the approximate primes less than any preassigned number. Riemann contributed significantly to the foundations of complex function theory, trigonometric series and consequently the development of integrals, etc.

[4] Let $(m/n)^2=2$ be such that m and n have no common factors. Then $m^2= 2\ n^2$ and hence m^2, consequently m, is even. Thus, if $m=2k$, then $n^2=2k^2$. Therefore n is even and so 2 is a common factor of m and n, which is a contradiction.

Given a gap on the real line, one can by a bisection process, confine the gap between two rationals, which are arbitrarily close to each other.

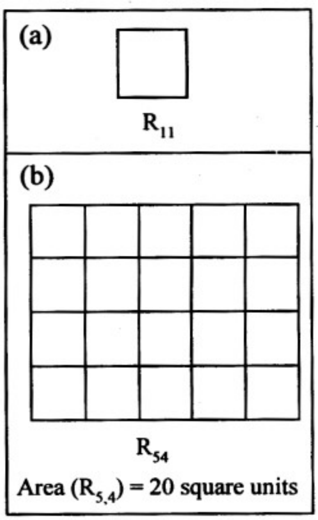
In mathematical literature, the gaps remaining on the line after the construction of rational points are known as 'irrational' points and the numbers corresponding to them are called 'irrational' numbers. These two sets of numbers (rational and irrational) together constitute the 'Real Number System'. In conclusion, real numbers are in one to one correspondence with the set of points on the line further justifying the term completion.

3. Area of Planar Regions

In this section, we try to assign some 'natural' meaning to the area enclosed by simple geometrical figures, namely the polygonal regions (see *Figure* 6c). The starting point here is to assign 'one square unit' to the area of a square (the simplest planar region) of size 1 (*Figure* 4a).

Note : A square of size l means a square with sides of length l units, which we denote by $R_{l,l}$ and so $R_{1,1}$ is the unit square.

Figure 4.



Intuitively, measuring the area (region) enclosed by a curve reduces to that of filling the region by such unit squares. Once the previous axiom that a square of unit size has the area one square unit is accepted, the areas of all polygonal domains can be defined or determined as follows. This again involves an infinite process as we have discussed earlier. Let us see this geometrically. Consider a rectangle $R_{m,n}$ with sides of length m and n, where m and n are integers. Then, it is easy to visualize that exactly mn unit squares can fit in $R_{m,n}$ (see *Figure* 4b). Thus

$$\text{Area}(R_{m,n}) = mn \text{ square units.} \qquad (1)$$

On the other hand, a unit square can be divided into n^2 small squares of equal size $\frac{1}{n}$. Thus

$$\text{Area of a square of size } \frac{1}{n} = \frac{1}{n^2} \cdot \text{Area } (R_{11}) = \frac{1}{n^2}. \qquad (2)$$

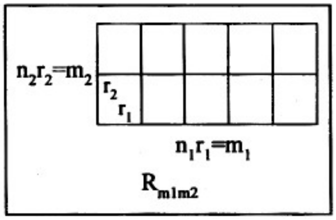


Figure 5.

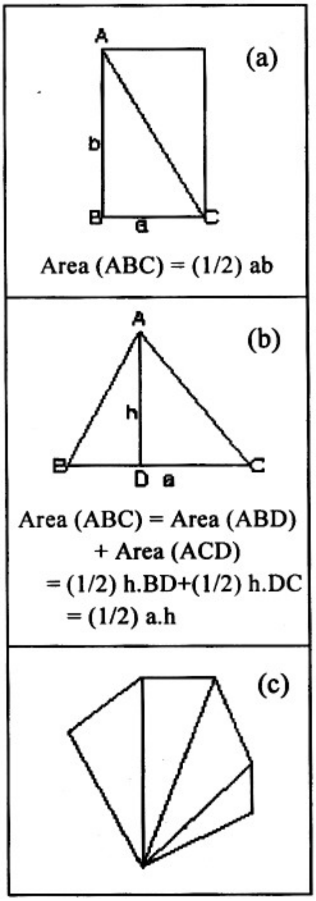
By putting together the above two arguments, that is the above two equations (1) and (2), we can get the area of a rectangle R_{r_1,r_2}, where $r_1 = \frac{m_1}{n_1}$, $r_2 = \frac{m_2}{n_2}$ are rational numbers. From the figure (*Figure* 5), it follows that

$$\text{Area}(R_{r_1,r_2}) = \frac{\text{Area}(R_{m_1,m_2})}{n_1 n_2} = \frac{m_1 m_2}{n_1 n_2} = r_1 r_2.$$

To find the area of a rectangle $R_{a,b}$ with sides of lengths a and b, where a, b are real numbers, one has to use the rational approximation. If a and b are represented respectively by rational sequences $\{r_n\}$ and $\{s_n\}$, then the rational sequence $\{r_n s_n\}$ represents a unique real number which corresponds to the product ab. Observe that the rational rectangle R_{r_n,s_n} approaches the rectangle $R_{a,b}$ as n approaches ∞. There are quite a few questions to be resolved to make the above statement unambiguous, but they are all part of the construction of the real number system. Once the construction is understood, the above mentioned fact is a trivial corollary.

But $\text{Area}(R_{r_n,s_n}) = r_n s_n$, which converges to ab. Thus, we have $\text{Area}(R_{a,b}) = ab$.

Figure 6.



Now it is an easy matter to see the areas of all polygonal domains. The areas of right angled triangles and then general triangles can be defined as in *Figures* 6a and b. Now observe that any polygonal domain can be divided into a finite number of triangles (*Figure* 6c) and hence the area is well defined.

Measuring the area enclosed by other curves is a more difficult task and involves an approximation process us-

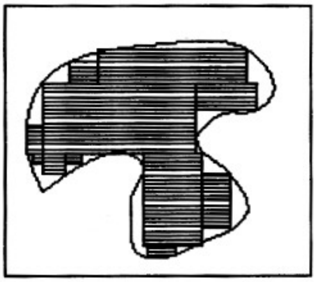


Figure 7.

[5] Henri-Léon Lebesgue was born on June 28, 1875 in Beauvais, Oise, Picardie, France. This French mathematician was famous for his generalization of R-integration which revolutionized the field of integration.

[6] To avoid technicalities, assume that f lies above the x-axis.

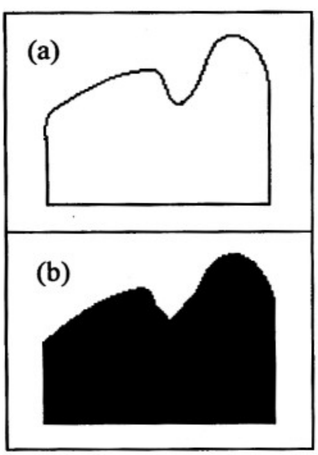
ing rectangles or polygons (see *Figure* 7). A systematic theory for this was developed and is known as 'integration'.

Three major mathematicians whose contributions to this are substantial are Augustin Cauchy, Bernhard Riemann (1826 - 1866) and Henri Lebesgue [5] (1875 - 1941). Cauchy proved the existence of an antiderivative of a continuous function constructing a sum based on the area which was later known as 'Cauchy Sum'. The Cauchy Sum was the starting point for Riemann as well and he generalized the concept to much more general functions and provided an integrability condition. His condition was so weak that the mathematical community once thought it to be the most general form of integration. But his theory, again was not powerful enough to handle many interesting and important mathematical problems. Later, Lebesgue further generalized the theory using 'measure theoretic' ideas. This led to 'measure' (the area/volume) of sets which are not necessarily bounded by nice curves/surfaces.

4. Integration (Cauchy)

We now introduce the concept of integration and the 'fundamental theorem of calculus'.

Figure 8.



Let f be a positive continuous function defined on an interval $[a, b], a, b$ being real numbers (see *Figure* 8a)[6]. The aim is to find the area of the region A (*Figure* 8b), enclosed by the graph of f, the *x-axis* and the two lines $x = a, x = b$.

Let $P : a = x_0 < x_1 < x_2 ... < x_n = b, n$ being an integer, be a partition of the interval $[a, b]$ and form the sum

$$S_P = \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i), \qquad (3)$$

where $t_i \in [x_{i-1}, x_i]$ be such that $f(t_i) = \text{Minimum } \{f(x) :$

$x \in [x_{i-1}, x_i]\}$. Since f is continuous in $[x_{i-1}, x_i]$, such a t_i can always be found. Then S_P is the area of the shaded region as shown in *Figure* 9a. Moreover, $S_P \leq$ Area(A) [7]. By adding more points to the partition P, we can get a new partition , say P', which we call a 'refinement' of P and then form the sum $S_{P'}$ similar to (3) (*Figure* 9b). It is a trivial matter to see that $S_P \leq S_{P'} \leq$ Area(A). So a refinement of a partition leads, possibly, to a better approximation of Area(A) or at least as good as the original partition. Hence we expect to cover the whole region A by taking more and more refinements, that is by an infinite process.

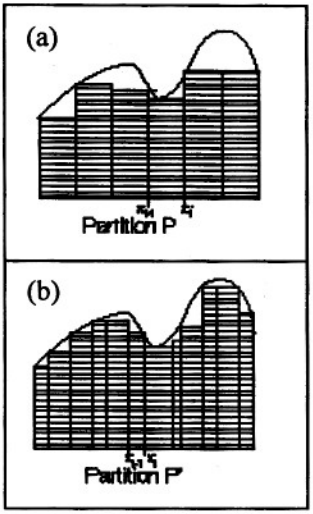


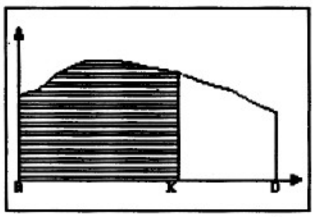
Figure 9.

Cauchy was the first person to prove that if f is continuous, then S_P becomes closer and closer to a unique real number, say k, as we take more and more refined partitions in such a way that $|P| :=$ Maximum $\{x_i - x_{i-1}, 1 \leq i \leq n\}$ becomes closer to zero. Such a limit will be independent of the partitions. Mathematically, we write $S_P \rightarrow k$ as $|P| \rightarrow 0$ (read as 'S_P converges to k as $mod(P)$ converges to zero'). The number k is the Area(A) and we call it the integral of f over a to b. Symbolically, $k = \int_a^b f(x)dx$ (read as 'integral of $f(x)dx$ from a to b'). The sum in (3) is known as 'Cauchy Sum'.

[7] This is due to the selection of t_i as above.

One need not chose only the value t_i at which f attains a minimum, but could chose any point in $[x_{i-1}, x_i]$ as far as continuous functions are concerned. For example, if we take points where the maximum is achieved, then we see that the area is approached from above and the limit is the same k as above. It is clear that if $f(x) = 0$ for all $a \leq x \leq b$, then $\int_a^t f(x)dx = 0$ for all $a \leq t \leq b$.

Figure 10.



In fact, Cauchy had done much more . Let t be any point in $[a, b]$ and $F(t)$ denote the area of the shaded region (see *Figure* 10) which can be determined as above. That is, $F(t) = \int_a^t f(x)dx$. Then Cauchy proved that the new function F is differentiable in $[a, b]$ and its derivative is given by f. i. e., $F'(t) = \lim_{h\rightarrow 0} \frac{F(t+h)-F(t)}{h} = f(t)$, proving that integration is indeed an 'inverse' process of differentiation. Thus F is an antiderivative[8] or primitive of f.

[8] A function H is called an 'antiderivative' or 'primitive' of f if $H'(t) = f(t)$ for all $a \leq t \leq b$.

If c is any constant and $G(x) = F(x) + c$, then $G' = F' = f$. That is, G is also a primitive of f. Conversely, it can be shown (by using the fact that if $f(x) = 0$ for all $a \leq x \leq b$, then $\int_a^t f(x)dx = 0$ for all $a \leq t \leq b$) that any primitive of f must be of the form $\int_a^x f(x)dx + C$. This leads to the following theorem.

Fundamental Theorem of Calculus

1. If f is continuous on $[a, b]$ and F is defined as above, then $F' = f$.

2. If G is such that $G'(x) = 0$ in $[a, b]$, then $G(x) =$ constant in $[a, b]$.

3. Any primitive of f must be of the form $\int_a^t f(x)dx + C, C$ is a constant. In other words, G is a function with continuous derivative G', then $\int_a^t G'(x)dx = G(t) - G(a)$.

In the next part of the article, we will present a quick overview of further developments in the integration theory without many details. We shall present an integrability condition due to Riemann for discontinuous functions.

Suggested Reading

[1] E T Bell, *Men of Mathematics*, Published by Simon I & Schuster, New York, 1965.

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Address for Correspondence
A K Nandakumaran
Department of Mathematics
Indian Institute of Science
Bangalore 560 012, India.
Email:
nands@math.iisc.ernet.in