MA 321 - ANALYSIS III (AUG-DEC, 2016)

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1. Show that $f = log|x|$ is locally integrable and hence defines a distribution. Calculate $\frac{d}{dx}log|x|$.

2. Define $< PV(\frac{1}{x})\ ,\ \phi >= \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\frac{\phi(x)}{x}dx$. Show that $PV(\frac{1}{x})$ defines a distribution and find $\frac{d}{dx}PV(\frac{1}{x})$.

3. Show that the following are distributions. Define $fp(\frac{1}{x^2})$ and $fp(\frac{H}{x^2})$ as

   $< fp(\frac{1}{x^2}),\ \ \phi >= \lim_{\epsilon\to 0}[\int\limits_{|x|\geq\epsilon}\frac{\phi(x)}{x^2}dx - 2\frac{\phi(0)}{\epsilon}]$ and

   $< fp(\frac{H}{x^2}),\ \ \phi >= \lim_{\epsilon\to 0}[\int\limits_{|x|\geq\epsilon}\frac{H(x)\phi(x)}{x^2}dx - \frac{\phi(0)}{\epsilon} + \phi'(0)log\epsilon]$

   for all $\phi\in\mathcal{D}(\mathbb{R})$, where H is the Heaviside function given by

   $H(x) = \begin{cases} 1 & \text{if } x\geq 0 \\ 0 & \text{otherwise}\end{cases}$

4. Let δ be the Dirac δ concentrated at 0. Show that there does not exists an L^2 function f such that $\delta = T_f$. In fact, δ cannot be identified with a locally integrable function. Similarly show that δ' cannot be given by a Radon measure.

5. Let f be a continuous function on $\mathbb{R}^n$ such that $\int f\phi = 0$ for all $\phi\in\mathcal{D}(\mathbb{R}^n)$. Show that $f\equiv 0$.

6. Let $T\in\mathcal{D}'(\mathbb{R})$ such that $\frac{dT}{dx} = 0$. Show that T is constant. (Hint: First show that, if $\phi\in\mathcal{D}(I\!R)$ then $\phi = \psi'$ for some $\psi\in\mathcal{D}(I\!R)$ if and only if $\int\phi(x)dx = 0$.)

7. Find all distribution solutions to the equation $\frac{dT}{dx} + aT = 0,\ \ a\in I\!R$.

8. Let $\delta(x) = C\exp(-\frac{1}{(x-1)^2})$ if $|x| < 1$, $= 0$ if $|x| > 1$. Find C so that $\displaystyle\int_{-1}^{1} \delta(x)dx = 1$. Define $\delta_n(x) = \epsilon^{-n}\delta(\frac{x}{\epsilon})$. Sketch δ_n and calculate $\displaystyle\int_{-1}^{1} \delta_n(x)dx$. Prove that if f is continuous, then $\displaystyle\int_{-\infty}^{\infty} f(x)\delta_n(x)dx \to f(0)$.

9. Show that $u(x,t) = f(x-kt)$, $f \in L^1_{loc}(I\!R)$ is a distribution solution of the differential equation

$$\frac{\partial^2 u}{\partial t^2} \; - \; k^2\,\frac{\partial^2 u}{\partial x^2} = 0$$

10. Show that $u(x_1, x_2) = log|x|^2$ is the distribution solution of

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 4\pi\delta.$$

11. Give an example of a sequence $\{\phi_j\}$ which converges to 0 in $C^\infty(\Omega)$, but not in $\mathcal{D}(\Omega)$.

12. Find the order and support of the following distributions

    1. $log|x|$, 2. $\delta^{(k)}$, 3. $P.V.(\frac{1}{x})$

    Give an example of a locally integrable function whose support is different from its support as a distribution.

13. Let $f \in C^1(\mathbb{R})$ except at finitely many points, say $a_1, \cdots, a_n$, having simple discontinuities. Denote DT_f and f' respectively, the distributional and classical derivatives of f. Assume f' is locally integrable. Show that

$$DT_f = T_{f'} + \Sigma_{i=1}^{n} J_f(a_i)\delta_{a_i}.$$

    where $J_f(a) = f(a+) - f(a-)$ is the jump at a and δ_{a_i} is the Dirac delta concentrated at a_i.

14. Let f be an absolutely continuous function on $\mathbb{R}$. Show that $T_{f'}$ is well defined $T_{f'} = DT_f$. (Hint: Use the integral $\displaystyle\int\int_A \phi'(x)f'(y)dxdy$ where $A = \{(x,y) \in \mathbb{R}^2 : -a \leq x < y \leq a\}$, where supp$\phi \subset [-a, a]$).

15. Construct a distribution of infinite order.