

Hausdorff operators in H^p spaces, $0 < p < 1$

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joint work with Akihiko Miyachi

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- In dimension one, after Kanjin, Miyachi, and Weisz, more or less final results were given in a joint paper by L-Miyachi.
- The results differ from those for L^p , $1 \leq p \leq \infty$, and H^1 , since they involve smoothness conditions on the averaging function, which seem unusual but unavoidable.

Definitions

Given a function ϕ on the half line $(0, \infty)$, the Hausdorff operator \mathcal{H}_ϕ is defined by

$$(\mathcal{H}_\phi f)(x) = \int_0^\infty \phi(t) t^{-1} f(t^{-1}x) dt, \quad x \in \mathbb{R}.$$

If $1 \leq p \leq \infty$, an application of Minkowski's inequality gives

$$\|\mathcal{H}_\phi f\|_{L^p(\mathbb{R})} \leq \int_0^\infty |\phi(t)| \|t^{-1} f(t^{-1}\cdot)\|_{L^p(\mathbb{R})} dt = A_p(\phi) \|f\|_{L^p(\mathbb{R})},$$

where

$$A_p(\phi) = \int_0^\infty |\phi(t)| t^{-1+1/p} dt.$$

Thus, \mathcal{H}_ϕ is bounded in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, provided that $A_p(\phi) < \infty$. Notice that the above simple argument for using Minkowski's inequality cannot be applied to $H^p(\mathbb{R})$ with $p < 1$.

We shall simply say that \mathcal{H}_ϕ is bounded in $H^p(\mathbb{R})$ if \mathcal{H}_ϕ is well-defined in a dense subspace of $H^p(\mathbb{R})$ and if it is extended to a bounded operator in $H^p(\mathbb{R})$.

- **Theorem A.** (Kanjin) Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Suppose $A_1(\phi) < \infty$, $A_2(\phi) < \infty$, and suppose $\widehat{\phi}$ (the Fourier transform of the function ϕ extended to the whole real line by setting $\phi(t) = 0$ for $t \leq 0$) is a function of class C^{2M} on \mathbb{R} with $\sup_{\xi \in \mathbb{R}} |\xi|^M |\widehat{\phi}^{(M)}(\xi)| < \infty$ and $\sup_{\xi \in \mathbb{R}} |\xi|^M |\widehat{\phi}^{(2M)}(\xi)| < \infty$. Then \mathcal{H}_ϕ is bounded in $H^p(\mathbb{R})$.

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- **Theorem B. (L-Miyachi)** Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and let ϵ be a positive real number. Suppose ϕ is a function of class C^M on $(0, \infty)$ such that

$$|\phi^{(k)}(t)| \leq \min\{t^\epsilon, t^{-\epsilon}\} t^{-1/p-k} \quad \text{for } k = 0, 1, \dots, M.$$

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Results

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- **Theorem C.** Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. If ϕ is a function on $(0, \infty)$ of class C^M and $\text{supp } \phi$ is a **compact** subset of $(0, \infty)$, then \mathcal{H}_ϕ is bounded in $H^p(\mathbb{R})$.

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- It is noteworthy that the above theorems impose certain smoothness assumption on ϕ . In fact, this smoothness assumption cannot be removed since we have the next theorem.
- **Theorem D. (L-Miyachi)** There exists a function ϕ on $(0, \infty)$ such that ϕ is bounded, $\text{supp } \phi$ is a compact subset of $(0, \infty)$, and, for every $p \in (0, 1)$, the operator \mathcal{H}_ϕ is not bounded in $H^p(\mathbb{R})$.

Special atomic decomposition - Miyachi

Definition. Let $0 < p \leq 1$ and let M be a positive integer. For $0 < s < \infty$, we define $\mathcal{A}_{p,M}(s)$ as the set of all those $f \in L^2(\mathbb{R}^n)$ for which $\widehat{f}(\xi) = 0$ for $|\xi| \leq \frac{1}{s}$ and

$$\|D^\alpha \widehat{f}\|_{L^2} \leq s^{|\alpha| - \frac{n}{p} + \frac{n}{2}}, \quad |\alpha| \leq M.$$

We define $\mathcal{A}_{p,M}$ as the union of $\mathcal{A}_{p,M}(s)$ over all $0 < s < \infty$.

Lemma. Let $0 < p \leq 1$ and M be a positive integer satisfying $M > \frac{n}{p} - \frac{n}{2}$. Then there exists a constant $c_{p,M}$, depending only on n , p and M , such that the following hold.

- (1) $\|f(\cdot - x_0)\|_{H^p(\mathbb{R}^n)} \leq c_{p,M}$ for all $f \in \mathcal{A}_{p,M}$ and all $x_0 \in \mathbb{R}^n$;
- (2) Every $f \in H^p(\mathbb{R}^n)$ can be decomposed as $f = \sum_{j=1}^{\infty} \lambda_j f_j(\cdot - x_j)$, where $f_j \in \mathcal{A}_{p,M}$, $x_j \in \mathbb{R}^n$, $0 \leq \lambda_j < \infty$, and

$$\left(\sum_{j=1}^{\infty} \lambda_j^p \right)^{\frac{1}{p}} \leq c_{p,M} \|f\|_{H^p(\mathbb{R}^n)}, \text{ and the series converges in } H^p(\mathbb{R}^n).$$

If $f \in H^p \cap L^2$, then this decomposition can be made so that the series converges in L^2 as well.

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$$\int_{\mathbb{R}^n} |u|^{-n} \Phi(u) f\left(\frac{x}{|u|}\right) du$$

is but indeed is not bounded in any $H^p(\mathbb{R}^n)$ with $p < 1$.

More general operators

Before proceeding to the multivariate case, consider a somewhat more advanced one-dimensional version of the Hausdorff operator, apparently first introduced by **Kuang**:

$$(\mathcal{H}f)(x) = (\mathcal{H}_{\varphi,a}f)(x) = \int_{\mathbb{R}_+} \frac{\varphi(t)}{a(t)} f\left(\frac{x}{a(t)}\right) dt,$$

where $a(t) > 0$ and $a'(t) > 0$ for all $t \in \mathbb{R}_+$ except maybe $t = 0$.

Denote $I_{\delta,T} = \{t : (1 - \delta)T \leq a(t) \leq T\}$, $0 < \delta < 1$.

Theorem E. Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and let ϵ be a positive real number. Suppose φ is a function of class C^M on $(0, \infty)$ such that φ and a satisfy the compatibility condition

$$\left| \left(\frac{1}{a'(t)} \frac{d}{dt} \right)^k \frac{\varphi(t)}{a'(t)} \right| \leq \min\{|a(t)|^\epsilon, |a(t)|^{-\epsilon}\} |a(t)|^{-1/p-k}$$

for $k = 0, 1, \dots, M$. Then $\mathcal{H}_{\varphi,a}$ is a bounded linear operator in H^p .

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- Let $N, n \in \mathbb{N}$, let $\Phi : \mathbb{R}^N \rightarrow \mathbb{C}$ and $A : \mathbb{R}^N \rightarrow M_n(\mathbb{R})$ be given, where $M_n(\mathbb{R})$ denotes the class of all $n \times n$ real matrices.

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- Assuming the matrix $A(u)$ be nonsingular for almost every u with $\Phi(u) \neq 0$, we define $\mathcal{H}_{\Phi,A}$ by

$$(\mathcal{H}_{\Phi,A}f)(x) = \int_{\mathbb{R}^N} \Phi(u) |\det A(u)|^{-1} f(x {}^tA(u)^{-1}) du, \quad x \in \mathbb{R}^n,$$

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- The Fourier transform of $\mathcal{H}_{\Phi,A}f$ is (formally) calculated from the definition as

$$(\mathcal{H}_{\Phi,A}f)^\wedge(\xi) = \int_{\mathbb{R}^N} \Phi(u) \widehat{f}(\xi A(u)) du, \quad \xi \in \mathbb{R}^n. \quad (1)$$

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- To be precise, we have to put some conditions on Φ , A , and f so that $\mathcal{H}_{\Phi,A}f$ is well-defined and the formula (1) holds.

Definitions

We give preliminary argument concerning the definition of $\mathcal{H}_{\Phi,A}$ and formula (1).

For functions $\Phi : \mathbb{R}^N \rightarrow \mathbb{C}$, $A : \mathbb{R}^N \rightarrow M_n(\mathbb{R})$, and $f : \mathbb{R}^n \rightarrow \mathbb{C}$, consider

$$(\mathcal{H}_{\Phi,A}f)(x) = \int_{\mathbb{R}^N} \Phi(u) |\det A(u)|^{-1} f(x {}^tA(u)^{-1}) du, \quad x \in \mathbb{R}^n,$$

and

$$(\tilde{\mathcal{H}}_{\Phi,A}f)(x) = \int_{\mathbb{R}^N} \Phi(u) f(xA(u)) du, \quad x \in \mathbb{R}^n.$$

We always assume that Φ , A , and f are Borel measurable functions.

Defining

$$L_A(\Phi) = \int_{\mathbb{R}^N} |\Phi(u)| |\det A(u)|^{-1/2} du,$$

we have the following.

Definitions

Proposition. If $L_A(\Phi) < \infty$, then for all $f \in L^2(\mathbb{R}^n)$ the functions $\mathcal{H}_{\Phi,A}f$ and $\tilde{\mathcal{H}}_{\Phi,A}f$ are well-defined almost everywhere on \mathbb{R}^n and the inequalities

$$\|\mathcal{H}_{\Phi,A}f\|_{L^2(\mathbb{R}^n)} \leq L_A(\Phi)\|f\|_{L^2(\mathbb{R}^n)}$$

and

$$\|\tilde{\mathcal{H}}_{\Phi,A}f\|_{L^2(\mathbb{R}^n)} \leq L_A(\Phi)\|f\|_{L^2(\mathbb{R}^n)}$$

hold. Thus $\mathcal{H}_{\Phi,A}$ and $\tilde{\mathcal{H}}_{\Phi,A}$ are well-defined bounded operators in $L^2(\mathbb{R}^n)$ if $L_A(\Phi) < \infty$.

The next proposition gives the formula (1).

Proposition. If $L_A(\Phi) < \infty$, then $(\mathcal{H}_{\Phi,A}f)^\wedge = \tilde{\mathcal{H}}_{\Phi,A}\hat{f}$ for all $f \in L^2(\mathbb{R}^n)$.

- On account of Theorems C and D, one may suppose that the multidimensional operator $\mathcal{H}_{\Phi,A}$ is bounded in $H^p(\mathbb{R}^n)$, $0 < p < 1$, if one merely assumes Φ and A to be sufficiently smooth and Φ to be with compact support.

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- However, this naive generalization of Theorem C is false. There are examples of smooth Φ with compact support and smooth A for which $\mathcal{H}_{\Phi,A}$ is not bounded in the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < 1$, $n \geq 2$.

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- This leads to conclusion that A , or Φ , or both of them should be subject to additional assumptions. The nature and type of such assumptions is, in a sense, the main issue, or, say, spirit of our work.

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- This leads to conclusion that A , or Φ , or both of them should be subject to additional assumptions. The nature and type of such assumptions is, in a sense, the main issue, or, say, spirit of our work.
- Indeed, for positive results, we introduce an algebraic condition on A and prove the Hardy space boundedness of $\mathcal{H}_{\Phi,A}$. This is a generalization of Theorem C to the multidimensional case.

Multidimensional result

Theorem. Let $n \in \mathbb{N}$, $n \geq 2$, $0 < p < 1$, and $M = [n/p - n/2] + 1$. Let $N \in \mathbb{N}$, $\Phi : \mathbb{R}^N \rightarrow \mathbb{C}$ be a function of class C^M with compact support, and $A : \mathbb{R}^N \rightarrow M_n(\mathbb{R})$ be a mapping of class C^{M+1} . Assume the matrix $A(u)$ is nonsingular for all $u \in \text{supp}\Phi$. Also assume Φ and A satisfy the following condition:

$$\left\{ \begin{array}{l} \text{for all } (u, y, \xi) \in \text{supp}\Phi \times \Sigma^{n-1} \times \Sigma^{n-1}, \\ \text{there exists a } j = j(u, y, \xi) \in \{1, \dots, N\} \text{ such that} \\ \left\langle y, \xi \frac{\partial A(u)}{\partial u_j} \right\rangle \neq 0, \end{array} \right. \quad (2)$$

where $\Sigma = \Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Then the operator $\mathcal{H}_{\Phi, A}$ is bounded in $H^p(\mathbb{R}^n)$.

Examples – unbounded

Example

Let Φ be a nonnegative smooth function on $(0, \infty)$ with compact support. Assume $\Phi(s) > 1$ for $1 < s < 2$. Then, for $n \geq 2$ and $0 < p < 1$, the operator $(Hf)(x) = \int_0^\infty \Phi(s)f(sx) ds$, $x \in \mathbb{R}^n$, is not bounded in $H^p(\mathbb{R}^n)$.

Let $SO(n, \mathbb{R})$ be the Lie group of real $n \times n$ orthogonal matrices with determinant 1 and let μ be the Haar measure on $SO(n, \mathbb{R})$.

Example

For $n \geq 2$ and $0 < p < 1$, the operator

$$(Hf)(x) = \int_{SO(n, \mathbb{R})} f(xP) d\mu(P), \quad x \in \mathbb{R}^n,$$

is not bounded in $H^p(\mathbb{R}^n)$.

Example – bounded

Example below should be compared with the preceding examples; the difference is only more dimensions but the result is quite opposite.

Example

Let $n \in \mathbb{N}$, $n \geq 2$, $0 < p < 1$, and $M = [n/p - n/2] + 1$. Let $\Phi : (0, \infty) \times SO(n, \mathbb{R}) \rightarrow \mathbb{C}$ be a function of class C^M with compact support. Then the operator

$$(Hf)(x) = \int_{(0, \infty) \times SO(n, \mathbb{R})} \Phi(s, P) f(sxP) ds d\mu(P), \quad x \in \mathbb{R}^n,$$

is bounded in $H^p(\mathbb{R}^n)$.

Dimensions

We give some remarks concerning the number N in the condition (2). To simplify notation, we write $B_j = \partial A(u)/\partial u_j$. Thus B_1, \dots, B_N are $n \times n$ real matrices. We consider the following condition:

$$\left\{ \begin{array}{l} \text{for all } (y, \xi) \in \Sigma^{n-1} \times \Sigma^{n-1}, \text{ there exists a } j \in \{1, \dots, N\} \\ \text{such that } \langle y, \xi B_j \rangle \neq 0. \end{array} \right. \quad (3)$$

We shall say that (3) is possible if there exist $B_1, \dots, B_N \in M_n(\mathbb{R})$ which satisfy (3).

The following statement is valid.

- Proposition.** (1) *The condition (3) is possible only if $N \geq n$.*
(2) *If n is odd and $n \geq 3$, then (3) is possible only if $N \geq n + 1$.*
(3) *For all $n \geq 2$, the condition (3) is possible with $N = 1 + n(n - 1)/2$. If $n \geq 4$, then (3) is possible with an $N < 1 + n(n - 1)/2$.*

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