Unimodular Bilinear multipliers on L^p spaces

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  for $f \in C_c^{\infty}(\mathbb{R}^n)$.
- By the use of the Plancherel Theorem it is easy to see that T_m is bounded on $L^2(\mathbb{R}^n)$.
- For $p \neq 2$, we need some regularity on m for T_m to be bounded on $L^p(\mathbb{R}^n)$.

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- Hörmander proved that if ϕ is a C^2 smooth real-valued function and $\|e^{i\lambda\phi}\|_{M_p(\mathbb{R}^n)} = O(1)$ for $\lambda \in \mathbb{R}$ and $p \neq 2$, then ϕ is a linear function.

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- He also conjectured that the above result holds for $\phi \in C^1(\mathbb{R}^n)$. In 1994 V. Lebedev and A. Olevskii [4] settled this conjecture and proved the above result for $\phi \in C^1(\mathbb{R}^n)$.

- Let $m(\xi,\eta)$ be a bounded measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and (p,q,r), $0 < p,q,r \leq \infty$ be a triplet of exponents. Consider the bilinear operator M_m initially defined for functions f and g in a suitable dense class by

$$M_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta. \quad (1)$$

Bilinear Multiplier

- Let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and (p, q, r), $0 < p, q, r \leq \infty$ be a triplet of exponents. Consider the bilinear operator M_m initially defined for functions f and g in a suitable dense class by

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- We say that M_m is a bilinear multiplier operator for the triplet (p, q, r) if M_m extends to a bounded operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, more precisely

$$\|M_m(f, g)\|_{L^r(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

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where C is independent of f and g.
- Let $\mathcal{M}_{p,q,r}(\mathbb{R}^n)$ be the space of all bilinear multiplier symbols for the triplet (p,q,r). The norm of $m \in \mathcal{M}_{p,q,r}(\mathbb{R}^n)$ is defined to be the norm of the corresponding bilinear multiplier operator M_m from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, i.e. $\|m\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^n)} = \|M_m\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^r(\mathbb{R}^n)}$.

- Let ϕ be a measurable function defined on $\mathbb{R}^n$. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ consider the bilinear operator

$$B(f,g)(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta) e^{i\phi(\xi,\eta)} e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta \qquad (2)$$

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- These types of bilinear multipliers arise when we study the solution of the non-linear PDE

$$\begin{aligned} i\partial_t u(t,x) + P(D)u(t,x) &= |u(x)|^2 \\ u(0,x) &= f(x), \end{aligned}$$

  where $P(D)$ is a quadratic homogeneous function of $D = (\partial_{x_1}, \partial_{x_2}, \cdots \partial_{x_n})$. The solution of the above PDE is given by

$$u(t,x) = e^{itP(D)}f(x) + \int_0^t e^{i(s-t)P(D)}(u(s,.),\overline{u(s,.)})(x)ds.$$

  It is therefore natural to study the above bilinear operators.

Definition

(Local L^2 range) The sets of exponents (p, q, r) satisfying $2 \leq p, q, r' \leq \infty$ and the Hölder condition $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ is referred to as the local L^2 range of exponents in the context of bilinear multipliers. We shall use the notation L for this set.

- F. Bernicot and P. Germain proved the following theorem:

Theorem

[2] Let us assume that

$$\partial_\eta \partial_\xi \phi \neq 0$$

$(\partial_\eta^2 - \partial_\eta \partial_\xi)\phi \neq 0$ *and* $\partial_\xi^2 - \partial_\eta \partial_\xi \phi \neq 0$. *Then the bilinear oscillatory integral operator*

$$T_\lambda(f,g)(x) := \frac{1}{(2\pi)^{1/2}} \int \int \hat{f}(\xi)\hat{g}(\eta) e^{i\lambda\phi(\xi,\eta)} e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$$

satisfies the following boundedness: for all exponents p, q, r *in the local* L^2 *range, there exists a constant* $C = C(p,q,r,\phi,m)$ *such that for all* $\lambda \neq 0$

$$\|T_\lambda(f,g)\|_{L^{r'}} \leq C|\lambda|^{-\frac{1}{2}} \|f\|_{L^{p'}} \|g\|_{L^{q'}}.$$

- We study the boundedness of the operator

$$T_{\phi,\lambda}(f,g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)e^{i\lambda\phi(\xi-\eta)}e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

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- We prove the following theorem

Theorem

Let (p,q,r) be a triplet of exponents outside the local L^2-range and satisfy the Hölder condition. If ϕ is a $\mathcal{C}^1(\mathbb{R}^n)$ smooth real nonlinear function, then

$$\|e^{i\lambda\phi(\xi-\eta)}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^n)} \to \infty,\ \lambda\in\mathbb{R},\ |\lambda|\to\infty.$$

- The theorem follows from the following lemma:

Lemma

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Suppose that there are N cubes $Q_k \subset \mathbb{R}^n$, $k = 1, 2, \dots, N$ such that $\phi(t) = \langle \alpha_k, t\rangle + \beta_k$ for almost all $t \in Q_k$, the vectors α_k, $k = 1, 2, \dots, N$ are all distinct and $\beta_k \in \mathbb{R}^n$. Then for any unbounded sequence of real numbers $\{\lambda_m\}_{m\in\mathbb{N}}$ we have

$$\sup_{m\in\mathbb{N}} \|e^{i\lambda_m\phi(\xi-\eta)}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^n)} \geq N^{\gamma-\frac{1}{2}}, \tag{3}$$

where $\gamma = \max\{\frac{1}{p}, \frac{1}{q}, \frac{1}{r'}\}$.

- It is easy to check that the R.H.S. of the last inequality clearly blows up when the triplet (p, q, r) lies outside the local L^2 range with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

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- The lemma is based on the following proposition:

Proposition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ be an $N-$tuple of distinct vectors of $\mathbb{R}^n$ and $\rho > 0$ be a positive real number. For vector-valued function $f = (f_1, f_2, \ldots, f_N)$ and $g = (g_1, g_2, \ldots, g_N)$ consider the bilinear operator

$$S_{\alpha,\rho}(f, g)(x) = (f_1(\cdot + \rho\alpha_1)g_1(\cdot - \alpha_1),$$

$$f_2(\cdot + \rho\alpha_2)g_2(\cdot - \alpha_2), \ldots, f_N(\cdot + \rho\alpha_1)g_N(\cdot - \alpha_N))(x).$$

Then the norm of the operator satisfies the following

$$\|S_{\alpha,\rho}\|_{p,q,r} \geq \max\{N^{1/p-1/2}, N^{1/q-1/2}\}.$$

Counter Examples in Local L^2 region

- We discuss examples of nonlinear functions ϕ for which $e^{i\phi}$ does not give rise to bilinear multiplier for exponents in the local L^2-range. Therefore, we cannot expect to have a consistent positive result concerning bilinear multipliers of the form $e^{i\phi}$, where ϕ is a nonlinear function, even in local L^2 range.

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$$T(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)e^{2i\xi\cdot\eta}e^{ix\cdot(\xi+\eta)}d\xi d\eta.$$

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- Define

$$\langle T^{*,1}(h,g), f\rangle := \langle T(f,g), h\rangle \text{ and } \langle T^{*,2}(f,h), g\rangle := \langle T(f,g), h\rangle.$$

It is known that if T is bounded from $L^p \times L^q$ into L^r, then $T^{*,1}$ is bounded from $L^{r'} \times L^q$ into $L^{p'}$ and $T^{*,2}$ is bounded from $L^p \times L^{r'}$ into $L^{q'}$.

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  $\langle T^{*,1}(h,g),f\rangle := \langle T(f,g),h\rangle$ and $\langle T^{*,2}(f,h),g\rangle := \langle T(f,g),h\rangle$.

  It is known that if T is bounded from $L^p \times L^q$ into L^r, then $T^{*,1}$ is bounded from $L^{r'} \times L^q$ into $L^{p'}$ and $T^{*,2}$ is bounded from $L^p \times L^{r'}$ into $L^{q'}$.
- Using the adjoint operators one can easily conclude that $e^{2i(\xi+\eta,-\eta)} \in \mathcal{M}_{r',q,p'}(\mathbb{R}^n)$ and $e^{2i(-\xi,\eta+\xi)} \in \mathcal{M}_{p,r',q'}(\mathbb{R}^n)$ with norm exactly the same as that of $e^{2i\xi\cdot\eta}$.

- **Case II: Interior points.** This case follows in a similar fashion. We claim that the function $e^{i(|\xi|^2+|\eta|^2)}$ does not give rise to a bilinear multiplier for any point in the interior of the region ABC.

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- Using the symmetry of the function $e^{i(|\xi|^2+|\eta|^2)}$, it is enough to show that $e^{i(|\xi|^2+|\eta|^2)} \notin \mathcal{M}_{p,p,\frac{p}{2}}(\mathbb{R}^n)$ for $2 < p < 4$.

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THANK YOU !!!