On Fourier Multipliers on the Heisenberg groups

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Fourier multipliers on $\mathbb{R}^n$

Given a bounded measurable function $m(\xi)$ on $\mathbb{R}^n$ we can define a transformation $T_m$ by setting

$$(T_m f)(\xi) = m(\xi)\hat{f}(\xi), \quad f \in L^2(\mathbb{R}^n).$$

By Plancherel theorem, $T_m$ is bounded operator on $L^2(\mathbb{R}^n)$. 
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\[
\hat{(T_m f)}(\xi) = m(\xi)\hat{f}(\xi), \quad f \in L^2(\mathbb{R}^n).
\]

By Plancherel theorem, \( T_m \) is bounded operator on \( L^2(\mathbb{R}^n) \).

**Definition (Fourier multiplier)**

When \( T_m \) extends to \( L^p(\mathbb{R}^n) \) as a bounded operator we say that \( m \) (or equivalently \( T_m \)) is a **Fourier multiplier** for \( L^p(\mathbb{R}^n) \).
Theorem (Hörmander’s multiplier theorem)

Let \( k = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( m \) be of class \( C^k \) away from the origin. If for any \( \beta \in \mathbb{N}^n \) satisfying \( |\beta| < k \)

\[
\sup_{R} R^{|\beta| - \frac{n}{2}} \left( \int_{R^n} |D^\beta m(\xi)|^2 \chi_{\{R < |\xi| < 2R\}}(\xi) d\xi \right)^{\frac{1}{2}} < \infty,
\]

then \( m \) is a Fourier multiplier for \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \). In particular, if

\[
|D^\beta m(\xi)| \leq C|\xi|^{-|\beta|},
\]

then \( m \) is an \( L^p \)-multiplier, \( 1 < p < \infty \).
Let us consider the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$ equipped with the group operation

$$(z, t)(w, s) = (z + w, t + s + \frac{i}{2} \Im z \bar{w}).$$

$H^n$ is a two-step nilpotent Lie group whose center is $\{(0, t) : t \in \mathbb{R}\}$. 

Schrödinger representation:

For each $\lambda \in \mathbb{R} \setminus \{0\}$, define $\pi_\lambda(z, t) \phi(\xi) = e^{i\lambda t} e^{i\lambda (x . \xi + \frac{1}{2} x . y)} \phi(\xi + y)$ where $\phi \in L^2(\mathbb{R}^n)$ and $(z, t) = (x + iy, t) \in H^n$. 

Notation: $\pi_\lambda(z) := \pi_\lambda(z, 0)$. 

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$$\pi_\lambda(z, t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2} x \cdot y)} \phi(\xi + y)$$

where $\phi \in L^2(\mathbb{R}^n)$ and $(z, t) = (x + iy, t) \in H^n$.

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**Fourier multipliers on $H^n$**
The group Fourier transform of a function \( f \in L^1(H^n) \) is given by

\[
\hat{f}(\lambda) = \int_{H^n} f(z, t) \pi_\lambda(z, t) \, dz \, dt.
\]

If \( f^\lambda(z) = \int_{\mathbb{R}} f(z, t) e^{i\lambda t} \, dt \), then the group Fourier transform can be written as

\[
\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z) \pi_\lambda(z) \, dz,
\]
Weyl Transform

Definition

Weyl transform of a function \( f \) on \( L^1(\mathbb{C}^n) \) in the following way:

\[
W_\lambda(f) = \int_{\mathbb{C}^n} f(z) \pi_\lambda(z) \, dz.
\]

We have the following relation between group Fourier transform on the Heisenberg group and Weyl Transform

\[
\hat{f}(\lambda) = W_\lambda(f^\lambda), \quad f \in L^1(H^n).
\]
Notation: $S_2$ := Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ with the inner product $(T, S) = tr(TS^*)$. 

Plancherel Theorem for Weyl Transform:

For a given $g \in L^1 \cap L^2(\mathbb{C}^n)$, it can be shown that $W_\lambda(g)$ is a Hilbert-Schmidt operator satisfying

$$||g||_{L^2}^2 = \left(\frac{2\pi}{n}\right)^n ||W_\lambda(g)||_{HS}.$$ 

Plancherel Theorem for group Fourier Transform:

For any given $f \in L^1 \cap L^2(\mathcal{H}^n)$ and for any $\lambda \in \mathbb{R}^\ast$, $\hat{f}(\lambda)$ is also a Hilbert-Schmidt operator. The map $f \rightarrow \hat{f}(\lambda)$ extends as an isometric isomorphism from $L^2(\mathcal{H}^n)$ to $L^2(\mathbb{R}^\ast, S_2, (2\pi)^{-n-1} |\lambda|^n d\lambda)$ and the Plancherel theorem can be read as

$$||f||_{L^2(\mathcal{H}^n)}^2 = \int_{-\infty}^{\infty} ||\hat{f}(\lambda)||_{HS}^2 |\lambda|^n d\lambda.$$
Plancherel theorems

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In fact the map $g \rightarrow W_\lambda(g)$ can be extended as an isometric isomorphism from $L^2(\mathbb{C}^n)$ to $S_2$, the space of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. 

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$$\|f\|_{L^2(\mathbb{H}^n)}^2 = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|^2_{HS} |\lambda|^n d\lambda.$$
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Plancherel Theorem for Weyl Transform:
For a given \( g \in L^1 \cap L^2(\mathbb{C}^n) \), it can be shown that \( W_\lambda(g) \) is a Hilbert-Schmidt operator satisfying

\[
\|g\|^2_{L^2} = (2\pi)^n |\lambda|^n \|W_\lambda(g)\|_{HS}.
\]

In fact the map \( g \to W_\lambda(g) \) can be extended as an isometric isomorphism from \( L^2(\mathbb{C}^n) \) to \( S_2 \), the space of all Hilbert-Schmidt operators on \( L^2(\mathbb{R}^n) \).

Plancherel Theorem for group Fourier Transform:
For any given \( f \in L^1 \cap L^2(H^n) \) and for any \( \lambda \in \mathbb{R}^* \), \( \hat{f}(\lambda) \) is also a Hilbert-Schmidt operator. The map \( f \to \hat{f}(\lambda) \) extends as an isometric isomorphism from \( L^2(H^n) \) to \( L^2(\mathbb{R}^*, S_2, (2\pi)^{-n-1} |\lambda|^n d\lambda) \) and the Plancherel theorem can be read as

\[
\|f\|^2_{L^2(H^n)} = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|^2_{HS} |\lambda|^n d\lambda.
\]
Definition (Weyl multipliers)

Given a bounded linear operator $m$ on $L^2(\mathbb{R}^n)$ we can define an operator $T^\lambda_m$ on $L^2(\mathbb{C}^n)$ by

$$W_\lambda(T^\lambda_m f) = mW_\lambda(f)$$

which is certainly bounded on $L^2(\mathbb{C}^n)$. If this operator extends to a bounded linear operator on $L^p(\mathbb{C}^n)$ then we say that $m$ is a (left) Weyl multiplier for $L^p(\mathbb{C}^n)$.

We can also define right Weyl multipliers.
Some notations and definitions

\[ A_j(\lambda) = \frac{\partial}{\partial \xi_j} + |\lambda| \xi_j, \quad A^*(\lambda) = -\frac{\partial}{\partial \xi_j} + |\lambda| \xi_j, \quad j = 0, 1, \ldots, n \]
Some notations and definitions

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2. \[ \delta_j(\lambda)m = |\lambda|^{-\frac{1}{2}} [m, A_j(\lambda)], \quad \bar{\delta}_j(\lambda)m = |\lambda|^{-\frac{1}{2}} [A^*(\lambda), m]. \]
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3. \( \delta^\alpha(\lambda) = \delta^\alpha_1(\lambda) \delta^\alpha_2(\lambda) \cdots \delta^\alpha_n(\lambda), \quad \bar{\delta}^\beta(\lambda) = \bar{\delta}^\beta_1(\lambda) \bar{\delta}^\beta_2(\lambda) \cdots \bar{\delta}^\beta_n(\lambda), \)

where \( \alpha, \beta \in \mathbb{N}^n \cup \{0\}. \)
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2. $\delta_j(\lambda)m = |\lambda|^{-\frac{1}{2}}[m, A_j(\lambda)]$, $\bar{\delta}_j(\lambda)m = |\lambda|^{-\frac{1}{2}}[A^*(\lambda), m]$.

3. $\delta^\alpha(\lambda) = \delta_1^{\alpha_1}(\lambda)\delta_2^{\alpha_2}(\lambda)\cdots\delta_n^{\alpha_n}(\lambda)$, $\bar{\delta}^\beta(\lambda) = \bar{\delta}_1^{\beta_1}(\lambda)\bar{\delta}_2^{\beta_2}(\lambda)\cdots\bar{\delta}_n^{\beta_n}(\lambda)$,

where $\alpha, \beta \in \mathbb{N}^n \cup \{0\}$.

4. We say that an operator $S \in B(L^2(\mathbb{R}^n))$ is of class $C^k$ if $\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)S \in B(L^2(\mathbb{R}^n))$ for all $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta| \leq k$. 

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Fourier multipliers on $H^n$
Some notations and definitions

1. \( A_j(\lambda) = \frac{\partial}{\partial \xi_j} + |\lambda| \xi_j, \quad A^*(\lambda) = -\frac{\partial}{\partial \xi_j} + |\lambda| \xi_j, \quad j = 0, 1, \ldots, n \)

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4. We say that an operator \( S \in B(L^2(\mathbb{R}^n)) \) is of class \( C^k \) if \( \delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)S \in B(L^2(\mathbb{R}^n)) \) for all \( \alpha, \beta \in \mathbb{N}^n \) such that \( |\alpha| + |\beta| \leq k. \)

5. \( \chi_k(\lambda) = \sum_{2^{k-1} \leq 2j + n < 2^k} P_j^\lambda \)

where \( P_j^\lambda \) are the projections onto the eigen space corresponding to the eigenvalue \( (2j + n)|\lambda| \) of the scaled Hermite operator \( H(\lambda) = -\Delta + |\lambda|^2|\chi|^2. \)
Theorem (Mauceri; J.Func. Anal; 1980)

Let $m \in B(L^2(\mathbb{R}^n))$ be an operator of class $C^{n+1}$ which satisfies the following conditions: For all $\alpha, \beta \in \mathbb{N}^n$, $|\alpha| + |\beta| \leq n + 1$

$$
\sup_{k \in \mathbb{N}^n} 2^k(|\alpha|+|\beta|-n)\|\left(\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m\right)\chi_k(\lambda)\|_{HS}^2 \leq C.
$$

Then the Weyl multiplier $T^\lambda_m$ is bounded on $L^p(\mathbb{C}^n)$, $1 < p \leq 2$ and is of weak type $(1, 1)$. 
Theorem (Mauceri; J.Func. Anal; 1980)

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\[
\sup_{k \in \mathbb{N}^n} 2^k(|\alpha| + |\beta| - n) \left\| (\delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda)m) \chi_k(\lambda) \right\|^2_{HS} \leq C.
\]

Then the Weyl multiplier \( T^\lambda_m \) is bounded on \( L^p(\mathbb{C}^n), 1 < p \leq 2 \) and is of weak type \((1,1)\). If the above assumption is replaced by

\[
\sup_{k \in \mathbb{N}^n} 2^k(|\alpha| + |\beta| - n) \left\| \chi_k(\lambda)(\delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda)m) \right\|^2_{HS} \leq C.
\]

Then the Weyl multiplier \( T^\lambda_m \) is bounded on \( L^p(\mathbb{C}^n), 2 \leq p < \infty \).
Theorem (S Bagchi, S Thangavelu, to be appear in J. Anal. Math)

Let \( m \in B(L^2(\mathbb{R}^n)) \) be an operator of class \( \mathcal{C}^{n+1} \) which satisfies the condition

\[
\sup_{k \in \mathbb{N}^n} 2^k(|\alpha|+|\beta|-n) \| (\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m) \chi_k(\lambda) \|_{HS}^2 \leq C .
\]

for all \( \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n + 1 \). Then the operator \( T^\lambda_m \) bounded on \( L^p(\mathbb{C}^n) \), \( 1 < p < \infty \).
Theorem (S Bagchi, S Thangavelu, to be appear in J. Anal. Math)

Let $m \in B(L^2(\mathbb{R}^n))$ be an operator of class $C^{n+1}$ which satisfies the condition

$$\sup_{k \in \mathbb{N}^n} 2^k(|\alpha|+|\beta|-n)\|(\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m)\chi_k(\lambda)\|_{HS}^2 \leq C.$$  

for all $\alpha, \beta \in \mathbb{N}^n$, $|\alpha| + |\beta| \leq n + 1$. Then the operator $T^\lambda_m$ bounded on $L^p(\mathbb{C}^n)$, $1 < p < \infty$. Moreover, $T^\lambda_m$ satisfies the weighted norm inequality

$$\int_{\mathbb{C}^n} |T^\lambda_m f(z)|^p w(z)dz \leq C(w) \int_{\mathbb{C}^n} |f(z)|^p w(z)dz$$

for all $w \in A_{p/2}(\mathbb{C}^n)$, $2 < p < \infty$. 

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Fourier multipliers on $H^n$
Corollary

Let \( \{ m(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^* \} \) be a family of operators satisfies the following inequality

\[
\sup_{k \in \mathbb{N}^n} 2^k(|\alpha|+|\beta|-n) \| (\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m(\lambda))\chi_k(\lambda) \|_{HS}^2 \leq C.
\]

for all \( \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n + 1 \). with uniform constant \( C \). Then we have the following vector-valued inequality

\[
\left\| \left( \sum_{i=1}^{n} |T_{m(\lambda_i)}f_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq \left\| \left( \sum_{i=1}^{n} |f_i|^2 \right)^{\frac{1}{2}} \right\|_p
\]

for any choice of \( \lambda_i \in \mathbb{R}^* \) and \( f_j \in L^p(\mathbb{C}^n) \).
Fourier Multipliers on $H^n$

**Definition**

Let $M = \{ M(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^* \}$ be a family of operators which are uniformly bounded. Then the operator $T_M$ is defined as follows

\[
(T_Mf)(\lambda) = M(\lambda)\hat{f}(\lambda).
\]

Here $\hat{f}$ stands for the group Fourier transform on the Heisenberg group. Clearly, $T_M$ is certainly bounded on $L^2(H^n)$. If this operator extends to a bounded linear operator on $L^p(H^n)$ then we say that $M$ is a (left) group Fourier multiplier for $L^p(H^n)$.

We can also define right Fourier multiplier for $L^p(H^n)$. 
Fourier multipliers were first studied by Mauceri and De-Michele (Michigan. Math. J) for $n = 1$ in 1979.

In 1995, Chin-Chen Lin (Rev. Math. Ibero) generalized their result for higher dimension.

The above result is very complicated and the proof is extremely technical and involves messy calculations.
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The above result is very complicated and the proof is extremely technical and involves messy calculations.

When $M(\lambda) = \varphi(H(\lambda))$, $H(\lambda)$ is the scaled Hermite operators on $\mathbb{R}^n$, the operator $T_M$ becomes $\varphi(\mathcal{L})$, where $\mathcal{L}$ is the sublaplacian of $H^n$. There are several works on the $L^p$ boundedness of $\varphi(\mathcal{L})$ and the best possible result has been obtained by Muller-Stein (J.Math. Pure. Appl, 1994) and Hebisch (Colloq. Math., 1993).
Notations and Definitions:

- For \( m \in \mathbb{N}^n \). Define

\[
\begin{align*}
  m_i^+ &= \max\{m_i, 0\}, \quad m_i^- = -\min\{m_i, 0\}, \\
  m^+ &= (m_1^+, m_2^+, \ldots, m_n^+), \quad m^- = (m_1^-, m_2^-, \ldots, m_n^-).
\end{align*}
\]
Notations and Definitions:

- For $m \in \bar{\mathbb{N}}^n$. Define
  \[
  m_i^+ = \max\{m_i, 0\}, \quad m_i^- = -\min\{m_i, 0\},
  \]
  \[
  m^+ = (m_1^+, m_2^+, \cdots, m_n^+), \quad m^- = (m_1^-, m_2^-, \cdots, m_n^-).
  \]

- Let $\Phi^\lambda_{\mu}, \mu \in \mathbb{N}^n$, are the scaled Hermite function. For \((\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \bar{\mathbb{N}}^n\), The partial isometries on $L^2(\mathbb{R}^n)$ can be defined as follows
  \[
  V_\alpha^m(\lambda)\Phi^\lambda_{\mu} = (-1)^{|m^+|}\delta_{\alpha+m^+,\mu}\Phi^\lambda_{\alpha+m^-}, \quad \text{when } \lambda > 0
  \]
  and
  \[
  V_\alpha^m(\lambda) = [V_\alpha^m(-\lambda)]^*, \quad \text{when } \lambda < 0.
  \]
  Here, $\delta_{\alpha,\beta}$ stands for kronecker delta.
If $M(\lambda) = \sum_{m,\alpha} B(\lambda, m, \alpha) V^m_\alpha(\lambda)$, then difference-differential operators are defined as follows: If $m_j \geq 1$,

$$
\Delta_{z_j} M(\lambda) = \left( \sum_{m,\alpha} B(\lambda, m - e_j, \alpha + e_j)(\alpha_j + 1)^{1/2} V^m_\alpha(\lambda) - \sum_{m,\alpha} B(\lambda, m - e_j, \alpha)(\alpha_j + m_j)^{1/2} V^m_\alpha(\lambda) \right),
$$

whereas if $m_j \leq 0$, then

$$
\Delta_{z_j} M(\lambda) = \sum_{m,\alpha} B(\lambda, m - e_j, \alpha - e_j) \alpha_j^{1/2} V^m_\alpha(\lambda) - \sum_{m,\alpha} B(\lambda, m - e_j, \alpha)(\alpha_j - m_j + 1)^{1/2} V^m_\alpha(\lambda).
$$
If $m_j \geq 1$,

$$
\Delta \bar{z}_j M(\lambda) = \left( \sum_{m,\alpha} B(\lambda, m + e_j, \alpha - e_j) \alpha_j^{\frac{1}{2}} V_\alpha^m(\lambda) \right) - \\
\sum_{m,\alpha} B(\lambda, m + e_j, \alpha)(\alpha_j + m_j + 1) \frac{1}{2} V_\alpha^m(\lambda),
$$

whereas if $m_j \leq 0$, then

$$
\Delta \bar{z}_j M(\lambda) = \sum_{m,\alpha} B(\lambda, m + e_j, \alpha + e_j)(\alpha_j + 1) \frac{1}{2} V_\alpha^m(\lambda) - \\
\sum_{m,\alpha} B(\lambda, m + e_j, \alpha)(\alpha_j - m_j) \frac{1}{2} V_\alpha^m(\lambda).
$$
\[ \Delta_t M(\lambda) = \sum_{m,\alpha} \frac{\partial}{\partial \lambda} B(\lambda, m, \alpha) V_\alpha^m(\lambda) + \frac{n}{2\lambda} \sum_{m,\alpha} B(\lambda, m, \alpha) V_\alpha^m(\lambda) + \]

\[ \frac{1}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^n \sqrt{\alpha_j (\alpha_j + m_j) B(\lambda, m, \alpha - e_j) V_\alpha^m(\lambda)} - \]

\[ \frac{1}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^n \sqrt{(\alpha_j + 1)(\alpha_j + m_j + 1) B(\lambda, m, \alpha + e_j) V_\alpha^m(\lambda)} \]
\( \Delta_t M(\lambda) = \sum_{m, \alpha} \frac{\partial}{\partial \lambda} B(\lambda, m, \alpha) V^m_{\alpha}(\lambda) + \frac{n}{2\lambda} \sum_{m, \alpha} B(\lambda, m, \alpha) V^m_{\alpha}(\lambda) + \)

\[
\frac{1}{2\lambda} \sum_{m, \alpha} \sum_{j=1}^n \sqrt{\alpha_j (\alpha_j + m_j) B(\lambda, m, \alpha - e_j) V^m_{\alpha}(\lambda)} -
\]

\[
\frac{1}{2\lambda} \sum_{m, \alpha} \sum_{j=1}^n \sqrt{(\alpha_j + 1)(\alpha_j + m_j + 1) B(\lambda, m, \alpha + e_j) V^m_{\alpha}(\lambda)}
\]

We can define \( \Delta_P \) for any polynomial \( P(z, \bar{z}, t) \) by using the definitions of \( \Delta_z, \Delta_{\bar{z}} \) and \( \Delta_t \).

We say a family of operators \( M = \{ M(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^* \} \) is in class \( C^k \) if \( \Delta_P M(\lambda) \in B(L^2(\mathbb{R}^n)) \) for every monomial \( P \) of degree \( \leq k \).
Theorem

Suppose $k \geq 4\left[\frac{n+5}{4}\right]$. Let $M$ be a family of operators which is in class $C^k$. Also assume

$$\sup_{\lambda \in \mathbb{R}^*} \|M(\lambda)\| \leq C$$

and

$$\sup_{k>0} 2^{k(\deg P - n - 1)} \int_{-\infty}^{\infty} \|[\Delta_P M(\lambda)] \chi_k(\lambda)\|_{HS}^2 |\lambda|^n d\lambda \leq C$$

for every monomial $P$ with $\deg P \leq 4\left[\frac{n+5}{4}\right]$. Then $T_M$ is $L^p$ bounded for $1 < p < \infty$ and also weak type $(1, 1)$. 
A family \( \{ m(\lambda) \in B(L^p(\mathbb{R}^n) : \lambda \in \mathbb{R} \} \) is called R-bounded if

\[
\left\| \left( \sum_{j=1}^{\infty} |m(\lambda_j)f_j|^2 \right)^{1/2} \right\|_p \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_p
\]

for all choices of \( \lambda_j \in \mathbb{R} \) and \( f_j \in L^p(\mathbb{R}^n) \).
A family \( \{ m(\lambda) \in B(L^p(\mathbb{R}^n)) : \lambda \in \mathbb{R} \} \) is called R-bounded if

\[
\| \left( \sum_{j=1}^{\infty} |m(\lambda_j)f_j|^2 \right)^{1/2} \|_p \leq C \| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \|_p
\]

for all choices of \( \lambda_j \in \mathbb{R} \) and \( f_j \in L^p(\mathbb{R}^n) \).

**Theorem (L. Weis)**

Let \( m(\lambda) \in B(L^p(\mathbb{R}^n)) \) for each \( \lambda \in \mathbb{R} \). Suppose \( \{ m(\lambda) : \lambda \in \mathbb{R} \} \) and \( \{ \lambda m'(\lambda) : \lambda \in \mathbb{R} \} \) are both R-bounded. Then the operator valued Fourier multiplier \( T_M \) defined by

\[
T_m f(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} m(\lambda) \hat{f}(\lambda) d\lambda
\]

extends to \( L^p(\mathbb{R}, L^p(\mathbb{R}^n)) \) as a bounded operator for all \( 1 < p < \infty \).
Let $M = \{ M(\lambda) \in B(L^2(R^n)) : \lambda \in \mathbb{R}^* \}$ be a family of operators. Suppose $T_M$ is the corresponding group Fourier multiplier. Also, let $T^\lambda_M(\lambda)$ be the Weyl multiplier associated to parameter $\lambda$ and operator $M(\lambda)$. Then one can easily show that

$$T_M f(z, t) = \int_{\mathbb{R}} e^{-i\lambda t} T^\lambda_M f^\lambda(z) d\lambda$$

for any $f \in L^2(H^n)$.

- We have already seen that if each $M(\lambda)$ satisfies Mauceri’s condition with uniform constant then $\{ M(\lambda) : \lambda \in \mathbb{R} \}$ is R-bounded.
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- $\{ \lambda M'(\lambda) : \lambda \in \mathbb{R} \}$ may not be R-bounded always. Example: Riesz transforms associated to the scaled Hermite operators.
A new version of Fourier multiplier theorem on $\mathcal{H}^n$

consider a new operator $\Theta(\lambda)$ defined as follows

$$\Theta(\lambda)m(\lambda) = \frac{d}{d\lambda}m(\lambda) - \frac{1}{2\lambda}[m(\lambda), \xi.\nabla] +$$

$$\frac{1}{2\lambda\sqrt{\lambda}} \sum_{j=1}^{n}(\delta_j(\lambda)m(\lambda)A_j^*(\lambda) + \delta_j^*(\lambda)m(\lambda)A_j(\lambda)).$$

If $g, tg \in L^2(\mathcal{H}^n)$, one can check that $(\hat{itg})(\lambda) = \Theta(\lambda)\hat{g}(\lambda)$. 
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An operator-valued function $M : \mathbb{R} \setminus \{0\} \rightarrow B(L^2(\mathbb{R}^n))$ is said to be in $E^k(\mathbb{R} \setminus \{0\})$ if $\delta^\alpha(\lambda)\delta^\beta(\lambda)\Theta^s(\lambda)$ are in $B(L^2(\mathbb{R}^n))$ for all $|\alpha| + |\beta| + 2s \leq k$ and $\lambda \in \mathbb{R} \setminus \{0\}$. 
Theorem (SB)

Let \( M \) be an operator-valued function which belongs to \( E^k(\mathbb{R} \setminus \{0\}) \), \( k \geq 2\left[\frac{n+3}{2}\right] \). Also, assume

\[
\sup_{\lambda \in \mathbb{R}\setminus\{0\}} \| M(\lambda) \| \leq C.
\]

If \( M \) satisfies

\[
\sup_{N > 0} 2^N (l-n-1) \int_{-\infty}^{\infty} \| \lambda - \frac{\alpha + \beta}{2} \delta^\alpha(\lambda) \delta^\beta(\lambda) \Theta^s(\lambda) M(\lambda) \chi_N(\lambda) \|_{HS}^2 |\lambda|^n d\lambda \leq C
\]

for all \( \alpha, \beta \in \mathbb{N}^n \), \( s \in \mathbb{N} \) satisfying \( |\alpha| + |\beta| + 2s = l \leq 2\left[\frac{n+3}{2}\right] \),

then \( T_M \) is weak type \((1,1)\) and bounded for \( 1 < p < \infty \).
Theorem (SB)

Let $M$ be an operator-valued function which belongs to $E^k(\mathbb{R} \setminus \{0\})$, $k \geq 2\left[\frac{n+3}{2}\right]$. Also, assume

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If $M$ satisfies

$$\sup_{N>0} 2^{N(l-n-1)} \int_{-\infty}^{\infty} \|\lambda - \frac{\alpha + \beta}{2} \delta^\alpha(\lambda) \overline{\delta^\beta(\lambda) \Theta^s(\lambda)} M(\lambda) \chi_N(\lambda)\|^2_{HS} |\lambda|^n d\lambda \leq C$$

for all $\alpha, \beta \in \mathbb{N}^n$, $s \in \mathbb{N}$ satisfying $|\alpha| + |\beta| + 2s = l \leq 2\left[\frac{n+3}{2}\right]$, then $T_M$ is weak type $(1, 1)$ and bounded for $1 < p < \infty$. Also,

$$\|T_M f\|_{L^p(w)} \leq C[w]^{\max\{1, \frac{1}{p-2}\}} \|f\|_{L^p(w)}$$

for all $w \in A_p^2(H^n)$, $2 < p < \infty$. 

Sayan Bagchi  
Fourier multipliers on $H^n$
Thank you