On Fourier Multipliers on the Heisenberg groups

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Fourier multipliers on $\mathbb{R}^n$

Given a bounded measurable function $m(\xi)$ on $\mathbb{R}^n$ we can define a transformation T_m by setting

$$\widehat{(T_m f)}(\xi) = m(\xi)\hat{f}(\xi), \qquad f \in L^2(\mathbb{R}^n).$$

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By Plancherel theorem, T_m is bounded operator on $L^2(\mathbb{R}^n)$.

Definition (Fourier multiplier)

When T_m extends to $L^p(\mathbb{R}^n)$ as a bounded operator we say that m (or equivalently T_m) is a Fourier multiplier for $L^p(\mathbb{R}^n)$.

Theorem (Hörmander's multiplier theorem)

Let $k = [\frac{n}{2}] + 1$ *and* m *be of class* C^k *away from the origin. If for any* $\beta \in \mathbb{N}^n$ *satisfying* $|\beta| < k$

$$\sup_R R^{|\beta| - \frac{n}{2}} \left(\int_{R^n} |D^\beta m(\xi)|^2 \chi_{\{R < |\xi| < 2R\}}(\xi) d\xi \right)^{\frac{1}{2}} < \infty,$$

then m *is a Fourier multiplier for* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$*. In particular, if*

$$|D^\beta m(\xi)| \leq C|\xi|^{-|\beta|},$$

then m *is an* L^p*-multiplier,* $1 < p < \infty$*.*

Heisenberg group

Let us consider the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$ equipped with the group operation

$$(z,t)(w,s) = (z+w, t+s+\frac{i}{2}\Im z.\bar{w}).$$

H^n is a two-step nilpotent Lie group whose center is $\{(0,t) : t \in \mathbb{R}\}$.

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Schödinger representation:

For each $\lambda \in \mathbb{R} \setminus \{0\}$, define

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x.\xi + \frac{1}{2}x.y)}\phi(\xi + y)$$

where $\phi \in L^2(\mathbb{R}^n)$ and $(z,t) = (x+iy, t) \in H^n$.

Notation: $\pi_\lambda(z) := \pi_\lambda(z,0)$.

Group Fourier Transform

Definition

The group Fourier transform of a function $f \in L^1(H^n)$ is given by

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

If $f^\lambda(z) = \int_{\mathbb{R}} f(z,t)e^{i\lambda t}dt$, then the group fourier transform can be written as

$$\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z)dz,$$

Weyl Transform

Definition

Weyl transform of a function f on $L^1(\mathbb{C}^n)$ in the following way:

$$W_\lambda(f) = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z)dz.$$

We have the following relation between group Fourier transform on the Heisenberg group and Weyl Transform

$$\hat{f}(\lambda) = W_\lambda(f^\lambda), \quad f \in L^1(H^n).$$

Plancherel theorems

Notation: $S_2 :=$ Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ with the inner product $(T, S) = tr(TS^*)$.

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Plancherel Theorem for Weyl Transform:

For a given $g \in L^1 \cap L^2(\mathbb{C}^n)$, it can be shown that $W_\lambda(g)$ is a Hilbert-Schmidt operator satisfying

$$||g||_{L^2}^2 = (2\pi)^n |\lambda|^n ||W_\lambda(g)||_{HS}.$$

In fact the map $g \to W_\lambda(g)$ can be extended as an isometric isomorphism from $L^2(\mathbb{C}^n)$ to S_2, the space of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$.

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Plancherel Theorem for group Fourier Transform:

For any given $f \in L^1 \cap L^2(H^n)$ and for any $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda)$ is also a Hilbert-Schmidt operator. The map $f \to \hat{f}(\lambda)$ extends as an isometric isomorphism from $L^2(H^n)$ to $L^2(\mathbb{R}^*, S_2, (2\pi)^{-n-1}|\lambda|^n d\lambda)$ and the Plancherel theorem can be read as

$$||f||_{L^2(H^n)}^2 = (2\pi)^{-n-1} \int_{-\infty}^{\infty} ||\hat{f}(\lambda)||_{HS}^2 |\lambda|^n d\lambda.$$

Definition (Weyl multipliers)

Given a bounded linear operator m on $L^2(\mathbb{R}^n)$ we can define an operator T_m^λ on $L^2(\mathbb{C}^n)$ by

$$W_\lambda(T_m^\lambda f) = mW_\lambda(f)$$

which is certainly bounded on $L^2(\mathbb{C}^n)$. If this operator extends to a bounded linear operator on $L^p(\mathbb{C}^n)$ then we say that m is a (left) Weyl multiplier for $L^p(\mathbb{C}^n)$.

We can also define right Weyl multipliers.

Some notations and definitions

1. $A_j(\lambda) = \frac{\partial}{\partial \xi_j} + |\lambda|\xi_j, \quad A^*(\lambda) = -\frac{\partial}{\partial \xi_j} + |\lambda|\xi_j, \quad j = 0, 1, \cdots, n$

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   where $\alpha, \beta \in \mathbb{N}^n \cup \{0\}$.

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   where $\alpha, \beta \in \mathbb{N}^n \cup \{0\}$.
4. We say that an operator $S \in B(L^2(\mathbb{R}^n))$ is of class C^k if $\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)S \in B(L^2(\mathbb{R}^n))$ for all $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta| \leq k$.

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5. $$\chi_k(\lambda) = \sum_{2^{k-1} \leq 2j+n < 2^k} P_j^\lambda$$
   where P_j^λ are the projections onto the eigen space corresponding to the eigenvalue $(2j + n)|\lambda|$ of the scaled Hermite operator $H(\lambda) = -\Delta + |\lambda|^2|x|^2$.

Theorem (Mauceri; J.Func. Anal; 1980)

Let $m \in B(L^2(\mathbb{R}^n))$ *be an operator of class* C^{n+1} *which satisfies the following conditions: For all* $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n+1$

$$\sup_{k \in \mathbb{N}^n} 2^{k(|\alpha|+|\beta|-n)} ||(\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m)\chi_k(\lambda)||^2_{HS} \leq C \ .$$

Then the Weyl multiplier T^λ_m *is bounded on* $L^p(\mathbb{C}^n), 1 < p \leq 2$ *and is of weak type* $(1,1)$.

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Then the Weyl multiplier T_m^λ is bounded on $L^p(\mathbb{C}^n), 1 < p \leq 2$ and is of weak type $(1,1)$. If the above assumption is replaced by

$$\sup_{k \in \mathbb{N}^n} 2^{k(|\alpha|+|\beta|-n)} ||\chi_k(\lambda)(\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m)||_{HS}^2 \leq C\ .$$

Then the Weyl multiplier T_m^λ is bounded on $L^p(\mathbb{C}^n), 2 \leq p < \infty$.

Theorem (S Bagchi, S Thangavelu, to be appear in J. Anal. Math)

Let $m \in B(L^2(\mathbb{R}^n))$ *be an operator of class* C^{n+1} *which satisfies the condition*

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for all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n+1.$ *Then the operator* T_m^λ *bounded on* $L^p(\mathbb{C}^n)$, $1 < p < \infty$.

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for all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n+1.$ *Then the operator* T_m^λ *bounded on* $L^p(\mathbb{C}^n)$*,* $1 < p < \infty$*.Moreover,* T_m^λ *satisfies the weighted norm inequality*

$$\int_{\mathbb{C}^n} |T_m^\lambda f(z)|^p w(z) dz \leq C(w) \ \int_{\mathbb{C}^n} |f(z)|^p w(z) dz$$

for all $w \in A_{p/2}(\mathbb{C}^n), 2 < p < \infty.$

Corollary

Let $\{m(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^*\}$ *be a family of operators satisfies the following inequality*

$$\sup_{k\in\mathbb{N}^n} 2^{k(|\alpha|+|\beta|-n)} ||(\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)m(\lambda))\chi_k(\lambda)||_{HS}^2 \leq C \ .$$

for all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n+1.$ *with uniform constant* C. *Then we have the following vector-valued inequality*

$$\| \left(\sum_{i=1}^{n} | T_{m(\lambda_i)}^{\lambda_i} f_i|^2 \right)^{\frac{1}{2}} \|_p \leq \| \left(\sum_{i=1}^{n} |f_i|^2 \right)^{\frac{1}{2}} \|_p$$

for any choice of $\lambda_i \in \mathbb{R}^*$ *and* $f_j \in L^p(\mathbb{C}^n)$.

Fourier Multipliers on H^n

Definition

Let $M = \{M(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^*\}$ *be a family of operators which are uniformly bounded. Then the operator* T_M *is defined as follows*

$$(\hat{T_M f})(\lambda) = M(\lambda)\hat{f}(\lambda).$$

Here $\hat{f}$ *stands for the group Fourier transform on the Heisenberg group. Clearly,* T_M *is certainly bounded on* $L^2(H^n)$*. If this operator extends to a bounded linear operator on* $L^p(H^n)$ *then we say that* M *is a (left) group Fourier multiplier for* $L^p(H^n)$*.*

We can also define right Fourier multiplier for $L^p(H^n)$.

- Fourier multipliers were first studied by Mauceri and De-Michele (Michigan. Math .J)for $n = 1$ in 1979.
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- In 1995, Chin-Chen Lin (Rev. Math. Ibero)generalized their result for higher dimension.
- The above result is very complicated and the proof is extremely technical and involves messy calculations.
- When $M(\lambda) = \varphi(H(\lambda))$, $H(\lambda)$ is the scaled Hermite operators on $\mathbb{R}^n$, the operator T_M becomes $\varphi(\mathcal{L})$, where $\mathcal{L}$ is the sublaplacian of H^n. There are several works on the L^p boundedness of $\varphi(\mathcal{L})$ and the best possible result has been obtained by Muller-Stein (J.Math. Pure. Appl, 1994) and Hebisch (Colloq. Math., 1993).

Notations and Definitions:

- For $m \in \bar{\mathbb{N}}^n$. Define

$$m_i^+ = max\{m_i, 0\}, \qquad m_i^- = -min\{m_i, 0\},$$

$$m^+ = (m_1^+, m_2^+, \cdots, m_n^+), \qquad m^- = (m_1^-, m_2^-, \cdots, m_n^-).$$

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- Let Φ_μ^λ, $\mu \in \mathbb{N}^n$, are the scaled Hermite function. For $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \bar{\mathbb{N}}^n$, The partial isometries on $L^2(\mathbb{R}^n)$ can be defined as follows

$$V_\alpha^m(\lambda)\Phi_\mu^\lambda = (-1)^{|m^+|}\delta_{\alpha+m^+,\mu}\Phi_{\alpha+m^-}^\lambda, \qquad \text{when } \lambda > 0$$

and

$$V_\alpha^m(\lambda) = [V_\alpha^m(-\lambda)]^*, \qquad \text{when } \lambda < 0.$$

Here, $\delta_{\alpha,\beta}$ stands for kronecker delta.

If $M(\lambda) = \sum_{m,\alpha} B(\lambda, m, \alpha) V_\alpha^m(\lambda)$, then difference-differential operators are defined as follows: If $m_j \geq 1$,

$$\Delta_{z_j} M(\lambda) = (\sum_{m,\alpha} B(\lambda, m - e_j, \alpha + e_j)(\alpha_j + 1)^{\frac{1}{2}} V_\alpha^m(\lambda) -$$

$$\sum_{m,\alpha} B(\lambda, m - e_j, \alpha)(\alpha_j + m_j)^{\frac{1}{2}} V_\alpha^m(\lambda),$$

whereas if $m_j \leq 0$, then

$$\Delta_{z_j} M(\lambda) = \sum_{m,\alpha} B(\lambda, m - e_j, \alpha - e_j)\alpha_j^{\frac{1}{2}} V_\alpha^m(\lambda) -$$

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If $m_j \geq 1$,

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$$\Delta_t M(\lambda) = \sum_{m,\alpha} \frac{\partial}{\partial\lambda} B(\lambda, m, \alpha) V_\alpha^m(\lambda) + \frac{n}{2\lambda} \sum_{m,\alpha} B(\lambda, m, \alpha) V_\alpha^m(\lambda) +$$

$$\frac{1}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^{n} \sqrt{\alpha_j(\alpha_j + m_j)} B(\lambda, m, \alpha - e_j) V_\alpha^m(\lambda) -$$

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We can define Δ_P for any polynomial $P(z, \bar{z}, t)$ by using the definitions of Δ_z, $\Delta_{\bar{z}}$ and Δ_t.
We say a family of operators $M = \{M(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^*\}$ is in class C^k if $\Delta_P M(\lambda) \in B(L^2(\mathbb{R}^n))$ for every monomial P of degree $\leq k$

Theorem

Suppose $k \geq 4[\frac{n+5}{4}]$. Let M be a family of operators which is in class C^k. Also assume

$$\sup_{\lambda \in \mathbb{R}^*} \|M(\lambda)\| \leq C$$

and

$$\sup_{k>0} 2^{k(degP-n-1)} \int_{-\infty}^{\infty} \|[\Delta_P M(\lambda)]\chi_k(\lambda)\|_{HS}^2 |\lambda|^n d\lambda \leq C$$

for every monomial P with $degP \leq 4[\frac{n+5}{4}]$. Then T_M is L^p bounded for $1 < p < \infty$ and also weak type $(1,1)$.

A approach using L. Weis's theorem

A family $\{m(\lambda) \in B(L^p(\mathbb{R}^n) : \lambda \in \mathbb{R}\}$ is called R-bounded if

$$\| \left(\sum_{j=1}^{\infty} |m(\lambda_j) f_j|^2 \right)^{1/2} \|_p \leq C \| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \|_p$$

for all choices of $\lambda_j \in \mathbb{R}$ and $f_j \in L^p(\mathbb{R}^n)$.

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for all choices of $\lambda_j \in \mathbb{R}$ and $f_j \in L^p(\mathbb{R}^n)$.

Theorem (L. Weis)

Let $m(\lambda) \in B(L^p(R^n))$ for each $\lambda \in \mathbb{R}$. Suppose $\{m(\lambda) : \lambda \in \mathbb{R}\}$ and $\{\lambda m'(\lambda) : \lambda \in \mathbb{R}\}$ are both R-bounded. Then the operator valued Fourier multiplier T_M defined by

$$T_m f(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} m(\lambda) \hat{f}(\lambda) d\lambda$$

extends to $L^p(\mathbb{R}, L^p(\mathbb{R}^n))$ as a bounded operator for all $1 < p < \infty$.

Let $M = \{M(\lambda) \in B(L^2(R^n)) : \lambda \in \mathbb{R}^*\}$ be a family of operators. Suppose T_M is the corresponding group Fourier multiplier. Also, let $T_M^\lambda(\lambda)$ be the Weyl multiplier associated to parameter λ and operator $M(\lambda)$. Then one can easily show that

$$T_M f(z,t) = \int_{\mathbb{R}} e^{-i\lambda t} T^\lambda_{M(\lambda)} f^\lambda(z) d\lambda$$

for any $f \in L^2(H^n)$.

- We have already seen that if each $M(\lambda)$ satisfies Mauceri's condition with uniform constant then $\{M(\lambda) : \lambda \in \mathbb{R}\}$ is R-bounded.

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- We have already seen that if each $M(\lambda)$ satisfies Mauceri's condition with uniform constant then $\{M(\lambda) : \lambda \in \mathbb{R}\}$ is R-bounded.
- $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$ may not be R-bounded always. Example: Riesz transforms associated to the scaled Hermite operators.

consider a new operator $\Theta(\lambda)$ defined as follows

$$\Theta(\lambda)m(\lambda) = \frac{d}{d\lambda}m(\lambda) - \frac{1}{2\lambda}[m(\lambda), \xi.\nabla] +$$

$$\frac{1}{2\lambda\sqrt{\lambda}}\sum_{j=1}^{n}(\delta_j(\lambda)m(\lambda)A_j^*(\lambda) + \delta_j^*(\lambda)m(\lambda)A_j(\lambda)).$$

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If $g, tg \in L^2(H^n)$, one can check that $\widehat{(itg)}(\lambda) = \Theta(\lambda)\hat{g}(\lambda)$.
An operator-valued function $M : \mathbb{R} \setminus \{0\} \to B(L^2(\mathbb{R}^n))$ is said to be in $E^k(\mathbb{R} \setminus \{0\})$ if $\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)\Theta^s(\lambda)$ are in $B(L^2(\mathbb{R}^n))$ for all $|\alpha| + |\beta| + 2s \leq k$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

Theorem (**SB**)

Let M be an operator-valued function which belongs to $E^k(\mathbb{R} \setminus \{0\})$, $k \geq 2[\frac{n+3}{2}]$. *Also, assume*

$$\sup_{\lambda \in \mathbb{R}\setminus\{0\}} \|M(\lambda)\| \leq C.$$

If M satisfies

$$\sup_{N>0} 2^{N(l-n-1)} \int_{-\infty}^{\infty} ||\lambda^{-\frac{\alpha+\beta}{2}} \delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda) \Theta^s(\lambda) M(\lambda) \chi_N(\lambda)||_{HS}^2 |\lambda|^n d\lambda \leq C$$

for all $\alpha, \beta \in \mathbb{N}^n$, $s \in \mathbb{N}$ *satisfying* $|\alpha| + |\beta| + 2s = l \leq 2[\frac{n+3}{2}]$, *then* T_M *is weak type* $(1,1)$ *and bounded for* $1 < p < \infty$.

Theorem (**SB**)

Let M be an operator-valued function which belongs to $E^k(\mathbb{R}\setminus\{0\})$, $k \geq 2[\frac{n+3}{2}]$. *Also, assume*

$$\sup_{\lambda\in\mathbb{R}\setminus\{0\}} \|M(\lambda)\| \leq C.$$

If M satisfies

$$\sup_{N>0} 2^{N(l-n-1)} \int_{-\infty}^{\infty} ||\lambda^{-\frac{\alpha+\beta}{2}}\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)\Theta^s(\lambda)M(\lambda)\chi_N(\lambda)||_{HS}^2 |\lambda|^n d\lambda \leq C$$

for all $\alpha, \beta \in \mathbb{N}^n$, $s \in \mathbb{N}$ *satisfying* $|\alpha| + |\beta| + 2s = l \leq 2[\frac{n+3}{2}]$, *then* T_M *is weak type* $(1,1)$ *and bounded for* $1 < p < \infty$. *Also,*

$$\|T_M f\|_{L^p(w)} \leq C[w]^{\max\{1,\frac{1}{p-2}\}}\|f\|_{L^p(w)}$$

for all $w \in A_{\frac{p}{2}}(H^n)$, $2 < p < \infty$.

Thank you