Hardy inequalities for fractional sublaplacians

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In honour of Professor S. Thangavelu
Indian Institute of Science, Bangalore, 21st December, 2017

Joint work with S. Thangavelu

Classical Hardy inequality

Due to G. H. Hardy. . .

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. . . but also contributions by E. Landau, G. Pólya, I. Schur, and M. Riesz (cf. A. Kufner, L. Maligranda, and L-E. Persson, *The prehistory of the Hardy inequality*, Amer. Math. Monthly) **113** (2006), 715–732

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A statement of Hardy inequality (Hardy 1920, 1925, also Hardy-Littlewood-Pólya, *Inequalities*):

Given f nonnegative (measurable) on $(0, \infty)$, then

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\, dt \right|^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\, dx$$

when $p > 1$ and RHS is finite

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Given f nonnegative (measurable) on $(0, \infty)$, let $F(x) = \int_0^x f(t)\, dt$, then

$$\int_0^\infty \left| \frac{F(x)}{x} \right|^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p \, dx$$

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Classical Hardy inequality (credits: M. J. Esteban)

Higher dimension $d \geq 3$, $p = 2$

$$0 \leq \int_{\mathbb{R}^d} \left| \nabla f + \alpha \frac{x}{|x|^2} f \right|^2 = \int_{\mathbb{R}^d} |\nabla f|^2 + \left(\alpha^2 - (d-2)\alpha\right) \int_{\mathbb{R}^d} \left| \frac{f(x)}{x} \right|^2$$

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and for all $\nu > \frac{(d-2)^2}{4}$ then $-\Delta - \frac{\nu}{|x|^2}$ is not bounded below

Fractional version of Hardy inequality

For $0 < s < d/2$, $f \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\, dx \leq C_{d,s} \langle \Delta^s f, f \rangle, \qquad C_{d,s} = 4^{-s} \frac{\Gamma(\frac{d-2s}{4})^2}{\Gamma(\frac{d+2s}{4})^2}$$

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- We may consider also a Hardy with a non-homogeneous potential

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x|^2)^s}\,dx \leq B_{d,s}\langle \Delta^s f, f\rangle$$

Hardy inequalities in the Heisenberg group

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  Key points - Strategy 2: Use the solutions of an *extension problem* (EP) for $\mathcal{L}$. This motivates study of EP in the Heisenberg group

The Heisenberg group $\mathbb{H}^d$: preliminaries

- $\mathbb{H}^d$ is the Lie group with underlying manifold $\mathbb{C}^d \times \mathbb{R}$ and multiplication

$$(z, w)(z', w') = \left(z + z', w + w' + \frac{1}{2}\,\mathrm{Im}(z \cdot \bar{z}')\right)$$

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$$\mathcal{L} := -\sum_j \Big(\big(\frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial w}\big)^2 + \big(\frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial w}\big)^2\Big) = -\sum_j (X_j^2 + Y_j^2)$$

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- Inverse FT in *the central variable* $\rightarrow f^\lambda(z) = \int_{-\infty}^{\infty} f(z, w) e^{i\lambda w} dw$

Spectral theory of the sublaplacian

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We have the expansion, for $f \in \mathbb{H}^d$,

$$f^\lambda(z) = c_d|\lambda|^d \sum_k \left(f^\lambda *_\lambda \varphi_k^\lambda(z)\right) \quad (*_\lambda \text{ is the } \textit{twisted convolution})$$

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- We will consider

$$\mathcal{L}_s := (2|T|)^s \frac{\Gamma\big(\frac{\mathcal{L}}{2|T|} + \frac{1+s}{2}\big)}{\Gamma\big(\frac{\mathcal{L}}{2|T|} + \frac{1-s}{2}\big)} \quad \to \text{ Fm: } \; (2|\lambda|)^s \frac{\Gamma(\frac{2k+d}{2} + \frac{1+s}{2})}{\Gamma(\frac{2k+d}{2} + \frac{1-s}{2})}$$

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3. Relation: $\mathcal{L}_s = U_s \mathcal{L}^s$, U_s bounded on $L^2(\mathbb{H}^d)$ (Stirling's formula)

The approach of the heat semigroup to define $\mathcal{L}_s$

- We have

$$e^{-t\mathcal{L}}f = f * q_t,$$

where

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- We use the numerical identity
$$\frac{\Gamma\left(\frac{\mu}{2}+\frac{1+s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1-s}{2}\right)} = c_s + C_s\int_0^\infty (1-e^{-\mu t})(\sinh t)^{-s-1}\,dt$$

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$$e^{-t\mathcal{L}}f = f * q_t,$$

where

$$\int_{-\infty}^{\infty} q_t(z,w)e^{i\lambda w}\,dw = |\lambda|^d \sum_{k=0}^{\infty} e^{-(2k+n)|\lambda|t}\varphi_k^\lambda(z) =: q_t^\lambda(z)$$

and $q_t^\lambda(z)$ has an explicit expression

- We use the numerical identity

$$\frac{\Gamma\left(\frac{2k+d}{2}+\frac{1+s}{2}\right)}{\Gamma\left(\frac{2k+d}{2}+\frac{1-s}{2}\right)} = c_s + C_s\int_0^\infty \left(1-e^{-(2k+d)t}\right)(\sinh t)^{-s-1}\,dt$$

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Theorem (Integral representation for $\mathcal{L}_s$)

For $0 < s < 1/2$

$$\mathcal{L}_s f(x) = \frac{1}{|\Gamma(-s)|} \int_{\mathbb{H}^d} \big(f(x) - f(y)\big) \mathcal{K}_s(y^{-1}x)\,dy, \quad f \in C_0^\infty(\mathbb{H}^d)$$

where $\mathcal{K}_s(z,w) = C_{d,s} |(z,w)|^{-2d-2-2s}$, and $C_{d,s} > 0$ is explicit

The extension problem for an operator L

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- Some references:
  - In the Euclidean case: Caffarelli-Silvestre (2007) (back to Molchanov-Ostrovskiĭ, DeBlassie. . .)
  - General second order differential operators: Stinga-Torrea (2010) (also Galé-Miana-Stinga, Banica-González. . .)
  - It occurs naturally in conformal geometry and scattering theory: Graham-Zworski, Chang-González. . .

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$$\text{(Our) proof: } v(z,w,\rho) = C_s \int_0^\infty \int_{\mathbb{R}} p_{t,s}(\rho,u)(e^{-t\mathcal{L}}\varphi)(z,w-u)\,du\,dt,$$

where $p_{t,s}$ is the heat kernel associated to the generalised sublaplacian

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Some remarks about the function $u_{s,\rho}(z,w)$:

- $u_{s,1}$ is connected with Poisson-type kernel when considering groups of type H
- $c_s u_{s,0}(z,w)$, **is the fundamental solution** of $\mathcal{L}_s$

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- Remark: All the results are shown for more general H-type groups

The operator $\mathcal{L}_s$ as a Dirichlet-to-Neumann operator

Let $v(z,w,\rho) = C_1(d,s)\rho^{2s}\varphi * u_{s,\rho}(z,w)$

Theorem

Assume φ and $\mathcal{L}_s\varphi$ are in $L^p(\mathbb{H}^d)$. Then

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ℓ is the integer such that $\ell - 1 \leq s < \ell$ and u is the solution of the EP. Then

$$\lim_{\rho\to 0} \rho^{2(\ell-s)}\Big(\frac{1}{2\rho}\partial_\rho\Big)^{\ell} u(z,w,\rho) = C(\ell,d,s)\mathcal{L}_s f(z,w),$$

in the L^p norm provided $\mathcal{L}_s f \in L^p(\mathbb{H}^d)$, where $C(\ell,d,s)$ is explicitly given

A generalised trace Hardy inequality in $\mathbb{H}^d$...

Define $\nabla = (X_1, \dots, X_d, Y_1, \dots, Y_d, \frac{1}{2}\rho\partial_w, \partial_\rho)$

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Let $\widetilde{W}_0^{s,2}(\mathbb{H}^d \times \mathbb{R}^+)$ be the completion of $C_0^\infty(\mathbb{H}^d \times \mathbb{R})$ with respect to the norm

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Theorem

Let $0 < s < 1$. For **any** *real valued function $u \in \widetilde{W}_0^{s,2}(\mathbb{H}^d \times \mathbb{R}^+)$, we have*

$$\int_0^\infty \int_{\mathbb{H}^d} |\nabla u(z,w,\rho)|^2 \rho^{1-2s}\, dz\, dw\, d\rho \geq \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)} \int_{\mathbb{H}^d} u^2(z,w,0) \frac{\mathcal{L}_s \varphi(z,w)}{\varphi(z,w)}\, dz\, dw$$

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Let $\mathbb{L} := -\mathcal{L} + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_t^2$; consider the extension problem

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- Use the previous theorem

. . . and a Hardy inequality

Corollary

$0 < s < 1$, $f, \mathcal{L}_s f \in L^2(\mathbb{H}^d)$. *Then, for* **any** $\varphi \in \operatorname{Dom} \mathcal{L}_s$, $\varphi^{-1}\mathcal{L}_s\varphi \in L^1_{\mathrm{loc}}(\mathbb{H}^d)$

$$(\mathcal{L}_s f, f) \geq \int_{\mathbb{H}^d} f^2(z,w) \frac{\mathcal{L}_s\varphi(z,w)}{\varphi(z,w)}\, dz\, dw$$

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Proof. Let $\mathcal{W}(z,w) = \frac{\mathcal{L}_s\varphi(z,w)}{\varphi(z,w)}$; by Cauchy–Schwarz

$$\Big(\int_{\mathbb{H}^d} |f(z,w)|^2\, dz\, dw\Big)^2 \leq \Big(\int_{\mathbb{H}^d} |f(z,w)|^2 \mathcal{W}(z,w)\, dz\, dw\Big)\Big(\int_{\mathbb{H}^d} |f(z,w)|^2 \mathcal{W}(z,w)^{-1}\, dz\, dw\Big)$$

Hardy inequality with non homogeneous weight

Recall $u_{s,\rho}(z,w) = ((\rho^2 + |z|^2)^2 + 16w^2)^{-\frac{d+s+1}{2}}$

Theorem (Cowling-Haagerup)

Let $\rho > 0$ *and* $0 < s < d+1$. *Then*

$$\mathcal{L}_s u_{-s,\rho}(z,w) = C_2(d,s)\rho^{2s} u_{s,\rho}(z,w),$$

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Theorem (Hardy inequality - non homogeneous)

Let $0 < s < 1$, $\delta > 0$, f, $\mathcal{L}_s f \in L^2(\mathbb{H}^d)$. Then

$$(\mathcal{L}_s f, f) \geq C_2(d,s)\delta^{2s} \int_{\mathbb{H}^d} \frac{f(z,w)^2}{\left((\delta^2 + |z|^2)^2 + 16w^2\right)^s} \, dz \, dw.$$

The (explicit) constant is sharp and equality is attained when $f = u_{-s,\delta}$

Hardy inequality with homogeneous weight

Theorem (Hardy inequality - homogeneous weight)

Let $0 < s < 1$*, and* f*,* $\mathcal{L}_s f \in L^2(\mathbb{H}^d)$*. Then*

$$(\mathcal{L}_s f, f) \geq C_2(d,s) \int_{\mathbb{H}^d} f^2(z,w) w_s(z,w)\, dz\, dw,$$

where w_s *is a function which is homogeneous of degree* $-2s$*.*
The explicit constant is sharp but equality is never achieved in $\widetilde{W}_0^{s,2}(\mathbb{H}^d)$

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- **Can we replace** $w_s(x)$ **by** $|x|^{-2s}$**?**

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- $w_s(x)$ is optimal in the following sense: If $\widetilde{w}_s \geq w_s$ is another weight s.t.

$$(\mathcal{L}_s f, f) \geq C_2(d,s) \int_{\mathbb{H}^d} f^2(z,w) \widetilde{w}_s(z,w)\, dz\, dw$$

  then $\widetilde{w}_s = w_s$

Thanks for your attention

Happy birthday Veluma!!

¡¡Que cumplas muchos más!!