Orthogonality and Approximation in Sobolev Spaces

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in honor of Professor Thangavelu
Let $\Omega$ be a regular domain in $\mathbb{R}^d$. Let $W^r_p(\Omega)$ denote a Sobolev space of $r$-th differentiable functions with norm $\| \cdot \|_{W^r_p(\Omega)}$.

Let $\Pi_n^d = \text{space of polynomials of degree } \leq n \text{ in } d\text{-variables}$.

**Problem:** For $f \in W^r_p(\Omega)$, find a polynomial $P_n \in \Pi_n^d$ such that

$$\| \partial^\alpha f - \partial^\alpha P_n \|_p \leq c n^{-r+|\alpha|} \| f \|_{W^r_p(\Omega)}, \quad |\alpha| \leq r,$$

where $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

**Question:** What $W^r_p(\Omega)$? How to construct such a $P_n$?

The problem is called *Simultaneous Approximation* in Approximation Theory. It arises recently from Spectral Method for numerical solution of PDE.
For \( f \in L^p[-1, 1], 1 \leq p \leq \infty \), define

\[
E_n(f)_p := \inf_{\deg P \leq n} \|f - P\|_p.
\]

A classical theorem states that, if \( f^{(r)} \in L^p[-1, 1] \), then

\[
E_n(f)_p \leq cn^{-r} \|\phi^r f^{(r)}\|_p, \quad \phi(x) = \sqrt{1 - x^2},
\]

for \( 1 \leq p \leq \infty \). The polynomial that attains this approximation order can be derived from Fourier-Legendre series,

\[
f \sim \sum_{n=1}^{\infty} \hat{f}_n P_n(x), \quad \hat{f}_n = \frac{\int_{-1}^{1} f(x) P_n(x) dx}{\int_{-1}^{1} [P_n(x)]^2 dx},
\]

where \( P_n \) is the Legendre polynomial of degree \( n \), satisfying

\[
\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{m,n}.
\]
For $p = 2$, the classical Hilbert space theory says that

$$E_n(f)_2 = \| f - S_n f \|_2,$$

where $S_n f = \sum_{k=1}^{n} \hat{f}_k P_k(x)$.

For $p \neq 2$, let $\eta$ be a smooth cut-off function on $\mathbb{R}$: $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \notin [0, 2]$; define

$$V_n f(x) := \sum_{k=0}^{2n} \eta \left( \frac{k}{n} \right) \hat{f}_k P_k(x).$$

Then $V_n f$ is of degree $2n$ and $V_n f = f$ if $f$ is a polynomial of degree $\leq n$, and

$$\| V_n f - f \|_p \leq cE_n(f)_p, \quad 1 \leq p \leq \infty.$$
If $f^{(r)} \in L^p[-1, 1]$, the simultaneous approximation asks to find a polynomial $P$, such that

$$\|f^{(k)} - P^{(k)}_n\|_p \leq cn^{-r+k}\|f^{(r)}\|_p, \quad k = 0, 1, \ldots, r$$

or an even stronger estimate

$$\|f^{(k)} - P^{(k)}_n\|_p \leq cn^{-r+k}E_{n-r}(f^{(r)})_p, \quad k = 0, 1, \ldots, r$$

The polynomial $P_n$, however, cannot come from the Fourier-Legendre series. In fact, for $p = 2$, we have

$$\left\|f^{(k)} - (S_nf)^{(k)}\right\|_2 \leq cn^{-r+2k-1/2}E_{n-r}(f^{(r)})_2, \quad k = 0, 1, \ldots, r$$

and the order on $n$ is sharp.
Approximation in Sobolev space with Jacobi weight

Let $w_{\alpha,\beta}(x) = (1 - x)^\alpha (1 + x)^\beta$, $\alpha, \beta > -1$, on $(-1, 1)$. The Jacobi polynomials $P_n^{(\alpha, \beta)}$ satisfy

$$\int_{-1}^{1} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) w_{\alpha,\beta}(x) dx = h_n^{\alpha, \beta} \delta_{n,m}.$$ 

All previous results extend to the space with $w_{\alpha,\beta}$. E.g.,

$$E_n(f)_{L^p(w_{\alpha,\beta})} := \inf_{p \in \Pi_n} \| f - p \|_{L^p(w_{\alpha,\beta})} \leq c n^{-r} \| \phi^r f(r) \|_{L^p(w_{\alpha,\beta})}.$$ 

The Fourier-Jacobi series of $f \in L^2(w_{\alpha,\beta})$ is defined by

$$f = \sum_{n=0}^{\infty} \hat{f}_n^{\alpha, \beta} P_n^{(\alpha, \beta)}, \quad \text{where} \quad \hat{f}_n^{\alpha, \beta} = \frac{\int_{-1}^{1} f(t) P_n^{(\alpha, \beta)}(t) w_{\alpha,\beta}(t) dt}{\int_{-1}^{1} [P_n^{(\alpha, \beta)}(t)]^2 w_{\alpha,\beta}(t) dt}$$

Let $S_n^{\alpha, \beta} f$ denote the $n$-th partial sum,

$$S_n^{\alpha, \beta} f(x) := \sum_{k=0}^{n} \hat{f}_k^{\alpha, \beta} P_k^{(\alpha, \beta)}(x).$$
Let $\partial = \frac{d}{dx}$. For $\alpha, \beta > -1$, the partial sum $S_{n}^{\alpha,\beta}$ satisfies

$$
\partial^{k} S_{n}^{\alpha,\beta} f = S_{n-k}^{\alpha+k,\beta+k} \partial^{k} f, \quad k = 0, 1, 2, \ldots
$$

It follows immediately that, for $0 \leq k \leq s$,

$$
\| \partial^{k} f - \partial^{k} S_{n}^{\alpha,\beta} f \|_{L^{2}(w_{\alpha+s,\beta+s})} \leq \| \partial^{k} f - S_{n-k}^{\alpha+k,\beta+k} f \|_{L^{2}(w_{\alpha+k,\beta+k})} = E_{n-k}(f^{(k)})_{L^{2}(w_{\alpha+k,\beta+k})} \leq cn^{-s+k} \| f^{(r)} \|_{L^{2}(w_{\alpha+s,\beta+s})}.
$$

Now, if we could choose $\alpha = \beta = -s$, this would be it.

However, $w_{\alpha,\beta}$ is NOT integrable if $a < -1$ and/or $\beta < -1$ and $P_{n}^{(-s,-s)}$ is NOT well defined for $s = 1, 2, \ldots$.

**Question:** Can we find an extension of $P_{n}^{(-s,-s)}$ AND an inner product to make the extension orthogonal?
Jacobi weight with negative indices

Jacobi polynomials of negative indices satisfy:

- \( \frac{d^s}{dx^s} P_n^{(-s,-s)} = c_n P_{n-s}^{(0,0)}(x). \)
- \( P_n^{(-s,-s)}(x) = (1 - x^2)^s P_{n-2s}^{(s,s)} \) for \( n \geq 2s \) (zero boundary).

**Spectral method community:** define

\[
\langle f, g \rangle = \int_{-1}^{1} f^{(s)}(x)g^{(s)}(x)dx
\]

work with functions that are have zero boundary conditions.

**Special function community:** true inner product (add boundary terms) but did not study Fourier orthogonal series.

The multiplication operator \( f \mapsto xf \) is no longer symmetric for Sobolev inner product. Consequently, NO three-term relation and NO Christopher-Darbox formula for the kernel – as in

\[
S_nf(x) = \int f(y)K_n(x, y)w(y)dy.
\]
Orthogonality in Sobolev space

For $s \in \mathbb{N}$ and $\theta \in [-1, 1]$, define an inner product in $W_2^s(w_{\alpha,\beta})$:

$$\langle f, g \rangle_{\alpha,\beta}^s := \int_{-1}^{1} f^{(s)}(t)g^{(s)}(t)w_{\alpha,\beta}(t)dt + \sum_{k=0}^{s-1} \lambda_k f^{(k)}(\theta)g^{(k)}(\theta),$$

where $\lambda_k$ are positive constants.

**Theorem**

*For $\alpha, \beta > -1$ and $s \in \mathbb{N}$. The polynomial

$$J_n^{\alpha-s, \beta-s}(x) := \begin{cases} 
\frac{(x - \theta)^n}{n!}, & 0 \leq n \leq s - 1, \\
\int_\theta^x \frac{(x - t)^{s-1}}{(s-1)!} P_{n-s}^\alpha(t)dt, & n \geq s.
\end{cases}$$

is orthogonal w.r.p. $\langle \cdot, \cdot \rangle_{\alpha,\beta}^s$ and its normal square satisfies

$$\mathcal{h}_{n}^{\alpha-s, \beta-s} = \lambda_n, \ 0 \leq n \leq s - 1, \ \text{and} \ \mathcal{h}_{n}^{\alpha-s, \beta-s} = h_{n-s}^{\alpha,\beta}, \ n \geq s.$$

Fourier expansion in Sobolev orthogonal polynomials

For $f \in W^s_p(w_{\alpha,\beta})$, consider the Fourier expansion

$$f = \sum_{n=0}^{\infty} \hat{f}_{n}^{\alpha-s,\beta-s} \mathcal{J}_{n}^{\alpha-s,\beta-s}$$

with

$$\hat{f}_{n}^{\alpha-s,\beta-s} := \frac{\langle f, \mathcal{J}_{n}^{\alpha-s,\beta-s} \rangle_{\alpha,\beta}}{\eta_{n}^{\alpha-s,\beta-s}}.$$

The $n$-th partial sum of this expansion is defined by

$$S_{n}^{\alpha-s,\beta-s} f := \sum_{k=0}^{n} \hat{f}_{k}^{\alpha-s,\beta-s} \mathcal{J}_{k,\theta}^{\alpha-s,\beta-s}.$$

**Lemma**

Let $\alpha, \beta > -1$ and $s \in \mathbb{N}$. For $f \in W^s_p(w_{\alpha,\beta})$ and $n = 0, 1, 2 \ldots$, with $m = \min\{n, s-1\}$,

1. $$S_{n}^{\alpha-s,\beta-s} f(x) = \sum_{k=0}^{m} f^{(k)}(\theta) \frac{(x-\theta)^k}{k!} + \int_{\theta}^{x} \frac{(x-t)^{s-1}}{(s-1)!} S_{n-s}^{\alpha,\beta} f(s)(t) \, dt,$$

2. $$\partial^{s} S_{n}^{\alpha-s,\beta-s} f = S_{n-s}^{\alpha,\beta} f(s) \text{ if } n \geq s.$$
Simultaneous approximation

Recall that we can define $V_n^{\alpha,\beta} f$ for $\alpha, \beta > -1$ via a cut-off function, good for approximation in $L^p$ norm. We can define

$$V_{n,\theta}^{\alpha-s,\beta-s} f(x) := \sum_{k=0}^{s-1} f^{(k)}(\theta) \frac{(x - \theta)^k}{k!} + \int_{\theta}^{x} \frac{(x - t)^{s-1}}{(s-1)!} V_{n-s}^{\alpha,\beta} f(s)(t) dt.$$ 

**Theorem**

Let $\alpha, \beta > -1$ and $f \in W^s_p(w_{\alpha,\beta})$ for $1 \leq p \leq \infty$. Then

$$\| \partial^k f - \partial^k V_{n,\theta}^{\alpha-s,\beta-s} f \|_{L^p(w_{\alpha,\beta})} \leq c n^{-s+k} E_n(f(s))_{L^p(w_{\alpha,\beta})}, \quad 0 \leq k \leq s,$$

if either $\theta = -1$ and $\beta = 0$ or $\theta = 1$ and $\alpha = 0$. Furthermore, for $p = 2$, we can replace $V_{n,\theta}^{\alpha-s,\beta-s} f$ by $S_{n,\theta}^{\alpha-s,\beta-s} f$. 
Let \( \varpi \) be a weight function define on \( \Omega \subset \mathbb{R}^d \). Let \( \mathcal{V}_n^d \) be the space of orthogonal polynomials of degree \( = n \) with respect to

\[
\langle f, g \rangle = \int_\Omega f(x)g(x)\varpi(x)dx.
\]

Let \( \{P_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^d\} \) be an orthogonal basis of \( \mathcal{V}_n^d \). Then \( \langle P_\alpha, P_\beta \rangle = 0, \alpha \neq \beta \) and \( \deg P_\alpha = n \). Let \( \hat{f}_n^\alpha := \langle f, P_\alpha^n \rangle / \langle P_\alpha^n, P_\alpha^n \rangle \).

- The projection operator \( \text{proj}_n : L^2 \mapsto \mathcal{V}_n^d \) and the \( n \)-th partial sum \( S_n : L^2 \mapsto \Pi_n^d \) are defined by

\[
\text{proj}_n f(x) := \sum_{|\alpha|=n} \hat{f}_\alpha P_\alpha(x), \quad S_n f(x) := \sum_{m=0}^n \text{proj}_m f(x).
\]

- The Fourier orthogonal expansion of \( f \in L^2 \) is defined by

\[
L^2(\Omega, \varpi) = \bigoplus_{n=0}^{\infty} \mathcal{V}_n^d, \quad f = \sum_{n=0}^{\infty} \text{proj}_n f
\]
Classical orthogonal polynomials on the unit ball

On the unit ball \( \mathbb{B}^d = \{ x : \| x \| \leq 1 \} \) of \( \mathbb{R}^d \), define
\[
\varpi_\mu(x) = (1 - \| x \|^2)^\mu, \quad \mu > -1.
\]
A basis for \( \mathcal{V}_n^d(\varpi_\mu) \) can be given via spherical harmonics. Let \( \mathcal{H}_n^d \) be the space of spherical harmonics of degree \( n \) in \( \mathbb{R}^d \).

**Theorem**

For \( n \in \mathbb{N}_0 \) and \( 0 \leq j \leq n/2 \), let \( \{ Y_{\ell}^{n-2j} : 1 \leq \ell \leq a_{n-2j}^d \} \) be an orthonormal basis for \( \mathcal{H}_{n-2j}^d \). Then
\[
P_{j,\ell}^{\mu,n}(x) := P_{j}^{(\mu,n-2j+\frac{d-2}{2})} (2 \| x \|^2 - 1) Y_{\ell}^{n-2j}(x).
\]

Then the set \( \{ P_{j,\ell}^{\mu,n}(x) : 1 \leq j \leq n/2, 1 \leq \ell \leq a_{n-2j}^d \} \) is an orthogonal basis of \( \mathcal{V}_n^d(\varpi_\mu) \).

Let \( \lambda_\mu = \mu + (d - 1)/2 \). These polynomials satisfy a PDE:
\[
(\Delta - \langle x, \nabla \rangle^2 - 2\lambda_\mu \langle x, \nabla \rangle)u = -n(n + \lambda_\mu)P, \quad \forall P \in \mathcal{V}_n^d(\varpi_\mu).
\]
Simultaneous approximation on the unit ball

We consider the unit ball $B^d = \{ x : \| x \| \leq 1 \}$ of $\mathbb{R}^d$ and the Sobolev space defined by

$$W^r_p(B^d) := \{ f \in L^p(B^d) : \partial^\alpha f \in L^p(B^d), \quad |\alpha| \leq r, \alpha \in \mathbb{N}_0^d \}$$

for $1 \leq p < \infty$ and by $C^r(B^d)$ if $p = \infty$. Its norm is defined by

$$\| f \|_{W^r_p(B^d)} := \left( \sum_{|\alpha| \leq r} \| \partial^\alpha f \|_{L^p(B^d)} \right)^{1/p}.$$

We hope to find a polynomial $P$ of degree $n$ such that for all derivatives up to $r$th order (Simultaneous Approximation)

$$\| \partial^\alpha f - \partial^\alpha P \|_p \leq cn^{-r+|\alpha|} \| f \|_{W^r_p(B^d)}, \quad |\alpha| \leq r.$$

This is established again by the Sobolev orthogonality. What we need resembles taking $\mu \to -s, \ s \in \mathbb{N}$, in the classical $\varpi_\mu$. 
Consider the Dirichlet problem for the Poisson equation:

\[-\Delta u = f \quad \text{in } \mathbb{B}^d \quad \text{with} \quad u = g \quad \text{on } S^{d-1}.

In variation form, we need to find \( u \in W^1_2(\mathbb{B}^d) \) such that

\[
\langle \nabla u, \nabla v \rangle_{\mathbb{B}^d} = \langle f, v \rangle_{\mathbb{B}^d} + d \langle g, v \rangle_{S^{d-1}}, \quad v \in W^1_2(\mathbb{B}^d),
\]

where

\[
\langle f, g \rangle_{\mathbb{B}^d} = \int_{\mathbb{B}^d} f(x)g(x)dx \quad \text{and} \quad \langle f, g \rangle_{S^{d-1}} = \int_{S^{d-1}} f(\xi)g(\xi)d\sigma.
\]

The Spectral Method looks for an approximate solution

\[
u_n = \sum_{j=1}^{N} a_j P^n_j,
\]

where \( \{P^n_j : 1 \leq j \leq N = \dim \Pi^d_n \} \) is a basis of \( \Pi^d_n \), such that

\[
\langle \nabla u_n, \nabla v \rangle_{\mathbb{B}^d} = \langle f, v \rangle_{\mathbb{B}^d} + d \langle g, v \rangle_{S^{d-1}}, \quad \text{for } v = ?.
\]
Orthogonality in Sobolev Space

The Galerkin method uses $v = P^n_j$ and determines the coefficients $a_j$, hence $u_n$, from the linear system

$$\sum_{j=1}^{N} a_j \langle \nabla P^n_j, \nabla P^n_k \rangle_{B^d} = \langle f, P^n_k \rangle_{B^d} + d \langle g, P^n_k \rangle_{S^{d-1}}, \quad 1 \leq k \leq N.$$

The matrix of the system becomes diagonal if $P^n_j$ are chosen as

$$\langle \nabla P^n_j, \nabla P^n_k \rangle_{B^d} = 0, \quad j \neq k.$$

This suggests that we consider the inner product

$$\langle f, g \rangle = \int_{B^d} \nabla f(x) \cdot \nabla g(x) \, dx + d \int_{S^{d-1}} f(\xi)g(\xi) \, d\sigma.$$

For the error estimate of this method, we could expect:

$$\| f - S_n f \|_{B^d} \leq c \, n^{-r} \| f \|_{W^r_2(B^d)}$$

$$\| \partial_i f - \partial_i S_n f \|_{B^d} \leq c \, n^{-r+1} \| f \|_{W^r_2(B^d)}, \quad 1 \leq i \leq d.$$
For $m = 1, 2, 3, \ldots$, let $\nabla^{2m} := \Delta^m$ and $\nabla^{2m+1} := \nabla \Delta^m$. For $\lambda_1, \ldots, \lambda_{\left\lceil \frac{s}{2} \right\rceil - 1} > 0$, define the inner product of $W^s_2(\mathbb{B}^d)$ by

$$\langle f, g \rangle_{-s} := \langle \nabla^s f, \nabla^s g \rangle_{\mathbb{B}^d} + \sum_{k=0}^{\left\lceil \frac{s}{2} \right\rceil - 1} \lambda_k \langle \Delta^k f, \Delta^k g \rangle_{\mathbb{S}^{d-1}}.$$

Let $\mathcal{V}_n^d(\mathcal{W}_{-s})$ denote the space of OP with respect to $\langle \cdot, \cdot \rangle_{-s}$.

- An orthogonal basis of $\mathcal{V}_n^d(\mathcal{W}_{-s})$ can be given explicitly. Denote these basis by $Q^{-s,n}_{j,\ell}(x)$, indexed again by $1 \leq \ell \leq a^d_{n-2j} = \dim \mathcal{H}^d_{n-2j}$ and $0 \leq j \leq n/2$ as $P_{j,\ell}^{\mu,n}$.

- $Q^{-s,n}_{j,\ell}$ are given in terms of Jacobi polynomials $P_n^{(-s,\beta)}$ and spherical harmonics, but the formulation is complicated.
Define the projection operator $\text{proj}_n^{-s} : W^s_2(\mathbb{B}^d) \rightarrow \mathcal{V}_n^d(\varpi_s)$ by

$$\text{proj}_n^{-s} f(x) := \sum_{0 \leq j \leq \frac{n}{2}} \sum_{\ell} \hat{f}_{j,\ell}^{-s,n} Q_{j,\ell}^{-s,n}(x), \quad \hat{f}_{j,\ell}^{-s,n} = \frac{\langle f, Q_{j,\ell}^{-s,n} \rangle_{-s}}{\langle Q_{j,\ell}^{-s,n}, Q_{j,\ell}^{-s,n} \rangle_{-s}}.$$

Unlike $[-1, 1]$, no nice formula for $\text{proj}_n^{-s} f$ is known for $d > 1$. Let $\text{proj}_n^0 : L^2(\mathbb{B}^d) \rightarrow \mathcal{V}_n^d$ be the classical operator ($\varpi_0(x) = 1$).

**Lemma**

*Let $n, s \in \mathbb{N}_0$ and $n \geq s$. If $s$ is even, then*

$$\Delta_2^s \text{proj}_n^{-s} f = \text{proj}_n^0 \Delta_2^s f.$$

*If $s$ is odd, then*

$$\partial_i \Delta_2^{\frac{s-1}{2}} \text{proj}_n^{-s} f = \text{proj}_n^0 \partial_i \Delta_2^{\frac{s-1}{2}} f, \quad i = 1, 2, \ldots, d.$$

This lemma is crucial for proving simultaneous approximation.
Simultaneous Approximation on the Ball

The partial sum and its analog defined via cut-off functions are

\[ S_n^{-s}f(x) := \sum_{k=0}^{n} \text{proj}_{n}^{-s} f(x), \quad V_n^{-s}f(x) := \sum_{k=0}^{2n} \eta \left( \frac{k}{n} \right) \text{proj}_{n}^{-s} f(x). \]

**Theorem**

Let \( r, s \in \mathbb{N} \) and \( r \geq s \). If \( f \in \mathcal{W}_p^r(\mathbb{B}^d) \), \( 1 < p < \infty \), then for \( n \geq s \)

\[ \| \partial^k f - \partial^k V_n^{-s} f \|_{L^p(\mathbb{B}^d)} \leq cn^{-r+k} \| f \|_{W_p^r(\mathbb{B}^d)}, \quad k = 0, 1, \ldots, s, \]

and \( V_n^{-s} f \) can be replaced by \( S_n^{-s} f \) if \( p = 2 \).

The proof relies on the Aubin-Nitsche duality argument in PDE, applied to a BVP of the equation \( \Delta^s u = v \) on \( \mathbb{B}^d \).

Simultaneous approximation on triangle

The study on triangle is far from complete. We state one result.

On the triangle $\triangle := \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$, define
\[ \partial_3 = \partial_2 - \partial_1. \]

We consider approximation in the norm of the Sobolev space
\[ \mathcal{W}_2^2 = \{ f \in L^2(\triangle) : \partial_i \partial_j f \in L^2(\triangle) : i, j = 1, 2 \}. \]

The classical weight function on the triangle is defined by
\[ \varpi_{\alpha, \beta, \gamma}(x, y) := x^\alpha y^\beta (1 - x - y)^\gamma, \quad \alpha, \beta, \gamma > -1. \]

Let $\mathcal{V}_n(\varpi_{\alpha, \beta, \gamma})$ = space of orthogonal polynomials of degree $n$. Several bases can be given via Jacobi polynomials. It is known
\[ \partial_1 : \mathcal{V}_n(\varpi_{\alpha, \beta, \gamma}) \mapsto \mathcal{V}_{n-1}(\varpi_{\alpha+1, \beta, \gamma+1}) \]
\[ \partial_2 : \mathcal{V}_n(\varpi_{\alpha, \beta, \gamma}) \mapsto \mathcal{V}_{n-1}(\varpi_{\alpha, \beta+1, \gamma+1}) \]
\[ \partial_3 : \mathcal{V}_n(\varpi_{\alpha, \beta, \gamma}) \mapsto \mathcal{V}_{n-1}(\varpi_{\alpha+1, \beta+1, \gamma}) \]
Sobolev orthogonality on triangle

For simultaneous approximation in $W^2_2$, we need to consider the Sobolev orthogonality in a new space $W^4_2$, defined by

$$W^4_2 = \{ f : \partial^2_1 \partial^2_2 \in L^2(\mathcal{W}_{0,0,2}), \partial^2_2 \partial^3_3 \in L^2(\mathcal{W}_{2,0,0}), \partial^2_3 \partial^2_1 \in L^2(\mathcal{W}_{0,2,0}) \}.$$ 

This amounts to:

- Define an inner product $\langle f, g \rangle_{-2,-2,-2}$ on the triangle (involving 4th order derivatives)
- Find a sequence of orthogonal polynomials on the triangle
- Let $S_{n}^{-2,-2,-2}$ be the $n$-th partial sum operator of the Fourier orthogonal expansion,

$$\begin{align*}
\partial^2_1 \partial^2_2 S_n^{0,0,2} f &= S_{n-4}^{0,0,2} \partial^2_1 \partial^2_2 f, \\
\partial^2_2 \partial^3_3 S_n^{0,2,0} f &= S_{n-4}^{0,2,0} \partial^2_2 \partial^3_3 f, \\
\partial^2_3 \partial^2_1 S_n^{2,0,0} f &= S_{n-4}^{2,0,0} \partial^2_3 \partial^2_1 f,
\end{align*}$$

where $S_n^{\alpha,\beta,\gamma} f$ in the RHS are partial sums for the Fourier series in classical orthogonal polynomials.
Simultaneous approximation on triangle

Let $E_n(f)_{\alpha, \beta, \gamma}$ denote the error of best approximation

$$E_n(f)_{\alpha, \beta, \gamma} := \inf_{P \in \Pi_n^2} \| f - P \|_{L^2(\mathcal{W}_{\alpha, \beta, \gamma})},$$

and write $E_n(f) = E_n(f)_{0,0,0}$. Define

$$\mathcal{E}_n(f) = E_n(\partial^2_1 \partial^2_2 f)_{0,0,2} + E_n(\partial^2_2 \partial^2_3 f)_{0,2,0} + E_n(\partial^2_3 \partial^2_1 f)_{2,0,0}.$$

**Theorem**

*For* $f \in \mathcal{W}^4_2$, *let* $p_n = S_n^{-2,-2,-2} f$. *Then*

\[
\begin{align*}
\| f - p_n \| & \leq \frac{c_1}{n^3} E_{n-3}(\partial_1 \partial_2 \partial_3) + \frac{c_2}{n^4} \mathcal{E}_{n-4}(f), \\
\| \partial_i f - \partial_i p_n \| & \leq \frac{c_1}{n^2} E_{n-3}(\partial_1 \partial_2 \partial_3) + \frac{c_2}{n^3} \mathcal{E}_{n-4}(f), \\
\| \partial_i \partial_j f - \partial_i \partial_j p_n \| & \leq \frac{c_1}{n} E_{n-3}(\partial_1 \partial_2 \partial_3) + \frac{c_2}{n^2} \mathcal{E}_{n-4}(f), \quad 1 \leq i, j \leq 3.
\end{align*}
\]

Xu, Constr. Approx. [2017]
Thank You

Happy 60, Thangavelu