Orthogonality and Approximation in Sobolev Spaces

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Approximation by polynomials in Sobolev space

Let Ω be a regular domain in $\mathbb{R}^d$. Let $W_p^r(\Omega)$ denote a Sobolev space of r-th differentiable functions with norm $\|\cdot\|_{W_p^r(\Omega)}$.

Let Π_n^d = space of polynomials of degree $\le n$ in d-variables.

Problem: For $f \in W_p^r(\Omega)$, find a polynomial $P_n \in \Pi_n^d$ such that

$$\|\partial^\alpha f - \partial^\alpha P_n\|_p \le c\, n^{-r+|\alpha|} \|f\|_{W_p^r(\Omega)}, \qquad |\alpha| \le r,$$

where $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Question: What $W_p^r(\Omega)$? How to construct such a P_n?

The problem is called *Simultaneous Approximation* in Approximation Theory. It arises recently from Spectral Method for numerical solution of PDE.

Approximation by polynomials on $[-1, 1]$

For $f \in L^p[-1,1]$, $1 \le p \le \infty$, define

$$E_n(f)_p := \inf_{\deg P \le n} \|f - P\|_p.$$

A classical theorem states that, if $f^{(r)} \in L^p[-1,1]$, then

$$E_n(f)_p \le cn^{-r} \|\phi^r f^{(r)}\|_p, \quad \phi(x) = \sqrt{1-x^2},$$

for $1 \le p \le \infty$. The polynomial that attains this approximation order can be derived from Fourier-Legendre series,

$$f \sim \sum_{n=1}^{\infty} \widehat{f}_n P_n(x), \qquad \widehat{f}_n = \frac{\int_{-1}^{1} f(x) P_n(x) dx}{\int_{-1}^{1} [P_n(x)]^2 dx},$$

where P_n is the Legendre polynomial of degree n, satisfying

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{m,n}.$$

Polynomial of approximation

For $p = 2$, the classical Hilbert space theory says that

$$E_n(f)_2 = \|f - S_n f\|_2, \quad \text{where} \quad S_n f = \sum_{k=1}^{n} \widehat{f}_k P_k(x).$$

For $p \neq 2$, let η be a smooth cut-off function on $\mathbb{R}$: $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \notin [0, 2]$; define

$$V_n f(x) := \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) \widehat{f}_k P_k(x).$$

Then $V_n f$ is of degree $2n$ and $V_n f = f$ if f is a polynomial of degree $\leq n$, and

$$\|V_n f - f\|_p \leq c E_n(f)_p, \qquad 1 \leq p \leq \infty.$$

Simultaneous Approximation on $[-1,1]$

If $f^{(r)} \in L^p[-1,1]$, the simultaneous approximation asks to find a polynomial P, such that

$$\left\|f^{(k)} - P_n^{(k)}\right\|_p \le c\, n^{-r+k} \|f^{(r)}\|_p, \quad k = 0, 1, \ldots, r$$

or an even stronger estimate

$$\left\|f^{(k)} - P_n^{(k)}\right\|_p \le c\, n^{-r+k} E_{n-r}(f^{(r)})_p, \quad k = 0, 1, \ldots, r$$

The polynomial P_n, however, cannot come from the Fourier-Legendre series. In fact, for $p = 2$, we have

$$\left\|f^{(k)} - (S_n f)^{(k)}\right\|_2 \le c\, n^{-r+2k-1/2} E_{n-r}(f^{(r)})_2, \quad k = 0, 1, \ldots, r$$

and the order on n is sharp.

Approximation in Sobolev space with Jacobi weight

Let $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, on $(-1,1)$. The Jacobi polynomials $P_n^{(\alpha,\beta)}$ satisfy

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)dx = h_n^{\alpha,\beta}\delta_{n,m}.$$

All previous results extend to the space with $w_{\alpha,\beta}$. E.g.,

$$E_n(f)_{L^p(w_{\alpha,\beta})} := \inf_{p\in\Pi_n} \|f-p\|_{L^p(w_{\alpha,\beta})} \le c\, n^{-r}\|\phi^r f^{(r)}\|_{L^p(w_{\alpha,\beta})}.$$

The Fourier-Jacobi series of $f \in L^2(w_{\alpha,\beta})$ is defined by

$$f = \sum_{n=0}^{\infty} \widehat{f}_n^{\alpha,\beta} P_n^{(\alpha,\beta)}, \quad \text{where} \quad \widehat{f}_n^{\alpha,\beta} = \frac{\int_{-1}^{1} f(t) P_n^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t)dt}{\int_{-1}^{1} [P_n^{(\alpha,\beta)}(t)]^2 w_{\alpha,\beta}(t)dt}$$

Let $S_n^{\alpha,\beta} f$ denote the n-th partial sum,

$$S_n^{\alpha,\beta} f(x) := \sum_{k=0}^{n} \widehat{f}_k^{\alpha,\beta} P_k^{(\alpha,\beta)}(x).$$

Jacobi weight with negative indices?

Let $\partial = \frac{d}{dx}$. For $\alpha, \beta > -1$, the partial sum $S_n^{\alpha,\beta}$ satisfies

$$\partial^k S_n^{\alpha,\beta} f = S_{n-k}^{\alpha+k,\beta+k} \partial^k f, \qquad k = 0, 1, 2, \ldots.$$

It follows immediately that, for $0 \le k \le s$,

$$\|\partial^k f - \partial^k S_n^{\alpha,\beta} f\|_{L^2(w_{\alpha+s,\beta+s})} \le \|\partial^k f - S_{n-k}^{\alpha+k,\beta+k} f\|_{L^2(w_{\alpha+k,\beta+k})}$$
$$= E_{n-k}(f^{(k)})_{L^2(w_{\alpha+k,\beta+k})} \le cn^{-s+k} \|f^{(r)}\|_{L^2(w_{\alpha+s,\beta+s})}.$$

Now, if we could choose $\alpha = \beta = -s$, this would be it.

However, $w_{\alpha,\beta}$ is NOT integrable if $a < -1$ and/or $\beta < -1$ and $P_n^{(-s,-s)}$ is NOT well defined for $s = 1, 2, \ldots$.

Question: Can we find an extension of $P_n^{(-s,-s)}$ AND an inner product to make the extension orthogonal?

Jacobi weight with negative indices

Jacobi polynomials of negative indices satisfy:

- $\frac{d^s}{dx^s} P_n^{(-s,-s)} = c_n P_{n-s}^{(0,0)}(x)$.
- $P_n^{(-s,-s)}(x) = (1-x^2)^s P_{n-2s}^{(s,s)}$ for $n \geq 2s$ (zero boundary).

Spectral method community: define

$$\langle f, g \rangle = \int_{-1}^{1} f^{(s)}(x) g^{(s)}(x) dx$$

work with functions that are have zero boundary conditions. Special function community: true inner product (add boundary terms) but did not study Fourier orthogonal series.

The multiplication operator $f \mapsto xf$ is no longer symmetric for Sobolev inner product. Consequently, NO three-term relation and NO Christopher-Darbox formula for the kernel – as in

$$S_n f(x) = \int f(y) K_n(x,y) w(y) dy.$$

Orthogonality in Sobolev space

For $s \in \mathbb{N}$ and $\theta \in [-1, 1]$, define an inner product in $W_2^s(w_{\alpha,\beta})$:

$$\langle f, g \rangle_{\alpha,\beta}^{-s} := \int_{-1}^{1} f^{(s)}(t) g^{(s)}(t) w_{\alpha,\beta}(t) dt + \sum_{k=0}^{s-1} \lambda_k f^{(k)}(\theta) g^{(k)}(\theta),$$

where λ_k are positive constants.

Theorem

For $\alpha, \beta > -1$ and $s \in \mathbb{N}$. The polynomial

$$\mathcal{J}_n^{\alpha-s,\beta-s}(x) := \begin{cases} \dfrac{(x-\theta)^n}{n!}, & 0 \le n \le s-1, \\ \displaystyle\int_\theta^x \frac{(x-t)^{s-1}}{(s-1)!} P_{n-s}^{\alpha,\beta}(t) dt, & n \ge s. \end{cases}$$

is orthogonal w.r.p. $\langle \cdot, \cdot \rangle_{\alpha,\beta}^{-s}$ and its normal square satisfies

$$\mathfrak{h}_n^{\alpha-s,\beta-s} = \lambda_n, \ 0 \le n \le s-1, \quad \text{and} \quad \mathfrak{h}_n^{\alpha-s,\beta-s} = h_{n-s}^{\alpha,\beta}, \ n \ge s.$$

Fourier expansion in Sobolev orthogonal polynomials

For $f \in W_p^s(w_{\alpha,\beta})$, consider the Fourier expansion

$$f = \sum_{n=0}^{\infty} \widehat{\mathfrak{f}}_n^{\alpha-s,\beta-s} \mathcal{J}_n^{\alpha-s,\beta-s} \quad \text{with} \quad \widehat{\mathfrak{f}}_n^{\alpha-s,\beta-s} := \frac{\langle f, \mathcal{J}_n^{\alpha-s,\beta-s} \rangle_{\alpha,\beta}^{-s}}{\mathfrak{h}_n^{\alpha-s,\beta-s}}.$$

The n-th partial sum of this expansion is defined by

$$\mathcal{S}_n^{\alpha-s,\beta-s} f := \sum_{k=0}^{n} \widehat{\mathfrak{f}}_k^{\alpha-s,\beta-s} \mathcal{J}_{k,\theta}^{\alpha-s,\beta-s}.$$

Lemma

Let $\alpha, \beta > -1$ and $s \in \mathbb{N}$. For $f \in W_p^s(w_{\alpha,\beta})$ and $n = 0, 1, 2 \ldots$,

1. *with $m = \min\{n, s-1\}$,*

$$\mathcal{S}_n^{\alpha-s,\beta-s} f(x) = \sum_{k=0}^{m} f^{(k)}(\theta) \frac{(x-\theta)^k}{k!} + \int_\theta^x \frac{(x-t)^{s-1}}{(s-1)!} S_{n-s}^{\alpha,\beta} f^{(s)}(t) dt,$$

2. $\partial^s \mathcal{S}_n^{\alpha-s,\beta-s} f = S_{n-s}^{\alpha,\beta} f^{(s)}$ *if $n \geq s$.*

Simultaneous approximation

Recall that we can define $V_n^{\alpha,\beta} f$ for $\alpha, \beta > -1$ via a cut-off function, good for approximation in L^p norm. We can define

$$\mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f(x) := \sum_{k=0}^{s-1} f^{(k)}(\theta) \frac{(x-\theta)^k}{k!} + \int_\theta^x \frac{(x-t)^{s-1}}{(s-1)!} V_{n-s}^{\alpha,\beta} f^{(s)}(t) dt.$$

Theorem

Let $\alpha, \beta > -1$ and $f \in W_p^s(w_{\alpha,\beta})$ for $1 \le p \le \infty$. Then

$$\|\partial^k f - \partial^k \mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f\|_{L^p(w_{\alpha,\beta})} \le c\, n^{-s+k} E_n(f^{(s)})_{L^p(w_{\alpha,\beta})}, \quad 0 \le k \le s,$$

if either $\theta = -1$ and $\beta = 0$ or $\theta = 1$ and $\alpha = 0$. Furthermore, for $p = 2$, we can replace $\mathcal{V}_{n,\theta}^{\alpha-s,\beta-s} f$ by $\mathcal{S}_{n,\theta}^{\alpha-s,\beta-s} f$.

Fourier orthogonal expansion in $\mathbb{R}^d$

Let ϖ be a weight function define on $\Omega \subset \mathbb{R}^d$. Let $\mathcal{V}_n^d$ be the space of orthogonal polynomials of degree = n with respect to

$$\langle f, g \rangle = \int_\Omega f(x) g(x) \varpi(x) dx.$$

Let $\{P_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^d\}$ be an orthogonal basis of $\mathcal{V}_n^d$. Then $\langle P_\alpha, P_\beta \rangle = 0$, $\alpha \neq \beta$ and $\deg P_\alpha = n$. Let $\widehat{f}_\alpha^n := \langle f, P_\alpha^n \rangle / \langle P_\alpha^n, P_\alpha^n \rangle$.

- The projection operator $\mathrm{proj}_n : L^2 \mapsto \mathcal{V}_n^d$ and the n-th partial sum $S_n : L^2 \mapsto \Pi_n^d$ are defined by

$$\mathrm{proj}_n f(x) := \sum_{|\alpha|=n} \widehat{f}_\alpha P_\alpha(x), \quad S_n f(x) := \sum_{m=0}^{n} \mathrm{proj}_m f(x).$$

- The Fourier orthogonal expansion of $f \in L^2$ is defined by

$$L^2(\Omega, \varpi) = \bigoplus_{n=0}^{\infty} \mathcal{V}_n^d, \qquad f = \sum_{n=0}^{\infty} \mathrm{proj}_n f$$

Classical orthogonal polynomials on the unit ball

On the unit ball $\mathbb{B}^d = \{x : \|x\| \le 1\}$ of $\mathbb{R}^d$, define

$$\varpi_\mu(x) = (1 - \|x\|^2)^\mu, \quad \mu > -1.$$

A basis for $\mathcal{V}_n^d(\varpi_\mu)$ can be given via spherical harmonics. Let $\mathcal{H}_n^d$ be the space of spherical harmonics of degree n in $\mathbb{R}^d$.

Theorem

For $n \in \mathbb{N}_0$ and $0 \le j \le n/2$, let $\{Y_\ell^{n-2j} : 1 \le \ell \le a_{n-2j}^d\}$ be an orthonormal basis for $\mathcal{H}_{n-2j}^d$. Then

$$P_{j,\ell}^{\mu,n}(x) := P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1)\, Y_\ell^{n-2j}(x).$$

Then the set $\{P_{j,\ell}^{\mu,n}(x) : 1 \le j \le n/2,\ 1 \le \ell \le a_{n-2j}^d\}$ is an orthogonal basis of $\mathcal{V}_n^d(\varpi_\mu)$.

Let $\lambda_\mu = \mu + (d-1)/2$. These polynomials satisfy a PDE:

$$(\Delta - \langle x, \nabla\rangle^2 - 2\lambda_\mu \langle x, \nabla\rangle)u = -n(n+\lambda_\mu)P, \quad \forall P \in \mathcal{V}_n^d(\varpi_\mu).$$

Simultaneous approximation on the unit ball

We consider the unit ball $\mathbb{B}^d = \{x : \|x\| \le 1\}$ of $\mathbb{R}^d$ and the Sobolev space defined by

$$W_p^r(\mathbb{B}^d) := \{f \in L^p(\mathbb{B}^d) : \partial^\alpha f \in L^p(\mathbb{B}^d), \quad |\alpha| \le r,\ \alpha \in \mathbb{N}_0^d\}$$

for $1 \le p < \infty$ and by $C^r(\mathbb{B}^d)$ if $p = \infty$. Its norm is defined by

$$\|f\|_{W_p^r(\mathbb{B}^d)} := \Big(\sum_{|\alpha| \le r} \|\partial^\alpha f\|_{L^p(\mathbb{B}^d)}\Big)^{1/p}.$$

We hope to find a polynomial P of degree n such that for all derivatives up to rth order (Simultaneous Approximation)

$$\|\partial^\alpha f - \partial^\alpha P\|_p \le c n^{-r+|\alpha|} \|f\|_{W_p^r(\mathbb{B}^d)}, \quad |\alpha| \le r.$$

This is established again by the Sobolev orthogonality. What we need resembles taking $\mu \to -s$, $s \in \mathbb{N}$, in the classical ϖ_μ.

Spectral method for Poisson equation

Consider the Dirichlet problem for the Poisson equation:

$$-\Delta u = f \quad \text{in } \mathbb{B}^d \quad \text{with} \quad u = g \quad \text{on } \mathbb{S}^{d-1}.$$

In variation form, we need to find $u \in W_2^1(\mathbb{B}^d)$ such that

$$\langle \nabla u, \nabla v \rangle_{\mathbb{B}^d} = \langle f, v \rangle_{\mathbb{B}^d} + d\langle g, v \rangle_{\mathbb{S}^{d-1}}, \quad v \in W_2^1(\mathbb{B}^d),$$

where

$$\langle f, g \rangle_{\mathbb{B}^d} = \int_{\mathbb{B}^d} f(x)g(x)dx \quad \text{and} \quad \langle f, g \rangle_{\mathbb{S}^{d-1}} = \int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\sigma.$$

The Spectral Method looks for an approximate solution

$$u_n = \sum_{j=1}^{N} a_j P_j^n,$$

where $\{P_j^n : 1 \le j \le N = \dim \Pi_n^d\}$ is a basis of Π_n^d, such that

$$\langle \nabla u_n, \nabla v \rangle_{\mathbb{B}^d} = \langle f, v \rangle_{\mathbb{B}^d} + d\langle g, v \rangle_{\mathbb{S}^{d-1}}, \quad \text{for } v = ?.$$

Orthogonality in Sobolev Space

The Galerkin method uses $v = P_j^n$ and determines the coefficients a_j, hence u_n, from the linear system

$$\sum_{j=1}^{N} a_j \langle \nabla P_j^n, \nabla P_k^n \rangle_{\mathbb{B}^d} = \langle f, P_k^n \rangle_{\mathbb{B}^d} + d \langle g, P_k^n \rangle_{\mathbb{S}^{d-1}}, \quad 1 \le k \le N.$$

The matrix of the system becomes diagonal if P_j^n are chosen as

$$\langle \nabla P_j^n, \nabla P_k^n \rangle_{\mathbb{B}^d} = 0, \qquad j \neq k.$$

This suggests that we consider the inner product

$$\langle f, g \rangle = \int_{\mathbb{B}^d} \nabla f(x) \cdot \nabla g(x) dx + d \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\sigma.$$

For the error estimate of this method, we could expect:

$$\|f - S_n f\|_{\mathbb{B}^d} \le c\, n^{-r} \|f\|_{W_2^r(\mathbb{B}^d)}$$
$$\|\partial_i f - \partial_i S_n\|_{\mathbb{B}^d} \le c\, n^{-r+1} \|f\|_{W_2^r(\mathbb{B}^d)}, \quad 1 \le i \le d.$$

Sobolev orthogonality on the unit ball

For $m = 1, 2, 3, ...$, let $\nabla^{2m} := \Delta^m$ and $\nabla^{2m+1} := \nabla\Delta^m$. For $\lambda_1, \ldots \lambda_{\lceil \frac{s}{2} \rceil - 1} > 0$, define the inner product of $W_2^s(\mathbb{B}^d)$ by

$$\langle f, g \rangle_{-s} := \langle \nabla^s f, \nabla^s g \rangle_{\mathbb{B}^d} + \sum_{k=0}^{\lceil \frac{s}{2} \rceil - 1} \lambda_k \langle \Delta^k f, \Delta^k g \rangle_{\mathbb{S}^{d-1}}.$$

Let $\mathcal{V}_n^d(\varpi_{-s})$ denote the space of OP with respect to $\langle \cdot, \cdot \rangle_{-s}$.

- An orthogonal basis of $\mathcal{V}_n^d(\varpi_{-s})$ can be given explicitly. Denote these basis by $Q_{j,\ell}^{-s,n}(x)$, indexed again by $1 \le \ell \le a_{n-2j}^d = \dim \mathcal{H}_{n-2j}^d$ and $0 \le j \le n/2$ as $P_{j,\ell}^{\mu,n}$.
- $Q_{j,\ell}^{-s,n}$ are given in terms of Jacobi polynomials $P_n^{(-s,\beta)}$ and spherical harmonics, but the formulation is complicated.

Orthogonal expansion in Sobolev space

Define the projection operator $\mathrm{proj}_n^{-s} : W_2^s(\mathbb{B}^d) \mapsto \mathcal{V}_n^d(\varpi_{-s})$ by

$$\mathrm{proj}_n^{-s} f(x) := \sum_{0 \le j \le \frac{n}{2}} \sum_{\ell} \widehat{f}_{j,\ell}^{-s,n} Q_{j,\ell}^{-s,n}(x), \quad \widehat{f}_{j,\ell}^{-s,n} = \frac{\left\langle f, Q_{j,\ell}^{-s,n} \right\rangle_{-s}}{\left\langle Q_{j,\ell}^{-s,n}, Q_{j,\ell}^{-s,n} \right\rangle_{-s}}.$$

Unlike $[-1,1]$, no nice formula for $\mathrm{proj}_n^{-s} f$ is known for $d > 1$.
Let $\mathrm{proj}_n^0 : L^2(\mathbb{B}^d) \mapsto \mathcal{V}_n^d$ be the classical operator ($\varpi_0(x) = 1$).

Lemma

Let $n, s \in \mathbb{N}_0$ and $n \ge s$. If s is even, then

$$\Delta^{\frac{s}{2}} \mathrm{proj}_n^{-s} f = \mathrm{proj}_{n-s}^0 \Delta^{\frac{s}{2}} f.$$

If s is odd, then

$$\partial_i \Delta^{\frac{s-1}{2}} \mathrm{proj}_n^{-s} f = \mathrm{proj}_{n-s}^0 \partial_i \Delta^{\frac{s-1}{2}} f, \quad i = 1, 2, \ldots, d.$$

This lemma is crucial for proving simultaneous approximation.

Simultaneous Approximation on the Ball

The partial sum and its analog defined via cut-off functions are

$$S_n^{-s} f(x) := \sum_{k=0}^{n} \operatorname{proj}_n^{-s} f(x), \quad V_n^{-s} f(x) := \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) \operatorname{proj}_n^{-s} f(x).$$

Theorem

Let $r, s \in \mathbb{N}$ and $r \geq s$. If $f \in W_p^r(\mathbb{B}^d)$, $1 < p < \infty$, then for $n \geq s$

$$\|\partial^k f - \partial^k V_n^{-s} f\|_{L_p(\mathbb{B}^d)} \leq c n^{-r+k} \|f\|_{W_p^r(\mathbb{B}^d)}, \quad k = 0, 1, \ldots, s,$$

and $V_n^{-s} f$ can be replaced by $S_n^{-s} f$ if $p = 2$.

The proof relies on the Aubin-Nitsche duality argument in PDE, applied to a BVP of the equation $\Delta^s u = v$ on $\mathbb{B}^d$.

Li-Xu, SIAM J. Numer. Anal. [2014].

Simultaneous approximation on triangle

The study on triangle is far from complete. We state one result.

On the triangle $\triangle := \{(x,y) : x \ge 0, y \ge 0, x+y \le 1\}$, define

$$\partial_3 = \partial_2 - \partial_1.$$

W consider approximation in the norm of the Sobolev space

$$W_2^2 = \{f \in L^2(\triangle) : \partial_i \partial_j f \in L^2(\triangle) : i,j = 1,2\}.$$

The classical weight function on the triangle is defined by

$$\varpi_{\alpha,\beta,\gamma}(x,y) := x^\alpha y^\beta (1-x-y)^\gamma, \qquad \alpha,\beta,\gamma > -1.$$

Let $\mathcal{V}_n(\varpi_{\alpha,\beta,\gamma})$ = space of orthogonal polynomials of degree n.
Several bases can be given via Jacobi polynomials. It is known

$$\partial_1 : \mathcal{V}_n(\varpi_{\alpha,\beta,\gamma}) \mapsto \mathcal{V}_{n-1}(\varpi_{\alpha+1,\beta,\gamma+1})$$
$$\partial_2 : \mathcal{V}_n(\varpi_{\alpha,\beta,\gamma}) \mapsto \mathcal{V}_{n-1}(\varpi_{\alpha,\beta+1,\gamma+1})$$
$$\partial_3 : \mathcal{V}_n(\varpi_{\alpha,\beta,\gamma}) \mapsto \mathcal{V}_{n-1}(\varpi_{\alpha+1,\beta+1,\gamma})$$

Sobolev orthogonality on triangle

For simultaneous approximation in W_2^2, we need to consider the Sobolev orthogonality in a new space $\mathcal{W}_2^4$, defined by

$$\mathcal{W}_2^4 = \{f : \partial_1^2\partial_2^2 \in L^2(\varpi_{0,0,2}), \partial_2^2\partial_3^2 \in L^2(\varpi_{2,0,0}), \partial_3^2\partial_1^2 \in L^2(\varpi_{0,2,0})\}.$$

This amounts to:

- Define an inner product $\langle f, g\rangle_{-2,-2,-2}$ on the triangle (involving 4th order derivatives)
- Find a sequence of orthogonal polynomials on the triangle
- Let $S_n^{-2,-2,-2}$ be the n-th partial sum operator of the Fourier orthogonal expansion,

$$\partial_1^2\partial_2^2 S_n^{-2,-2,-2} f = S_{n-4}^{0,0,2}\partial_1^2\partial_2^2 f,$$
$$\partial_2^2\partial_3^2 S_n^{-2,-2,-2} f = S_{n-4}^{0,2,0}\partial_2^2\partial_3^2 f,$$
$$\partial_3^2\partial_1^2 S_n^{-2,-2,-2} f = S_{n-4}^{2,0,0}\partial_3^2\partial_1^2 f,$$

where $S_n^{\alpha,\beta,\gamma} f$ in the RHS are partial sums for the Fourier series in classical orthogonal polynomials.

Simultaneous approximation on triangle

Let $E_n(f)_{\alpha,\beta,\gamma}$ denote the error of best approximation

$$E_n(f)_{\alpha,\beta,\gamma} := \inf_{p \in \Pi_n^2} \|f - P\|_{L^2(\varpi_{\alpha,\beta,\gamma})},$$

and write $E_n(f)_= E_n(f)_{0,0,0}$. Define

$$\mathcal{E}_n(f) = E_n(\partial_1^2 \partial_2^2 f)_{0,0,2} + E_n(\partial_2^2 \partial_3^2 f)_{0,2,0} + E_n(\partial_3^2 \partial_1^2 f)_{2,0,0}.$$

Theorem

For $f \in \mathcal{W}_2^4$, let $p_n = S_n^{-2,-2,-2} f$. Then

$$\begin{aligned} \|f - p_n\| &\le \frac{c_1}{n^3} E_{n-3}(\partial_1 \partial_2 \partial_3) + \frac{c_2}{n^4} \mathcal{E}_{n-4}(f), \\ \|\partial_i f - \partial_i p_n\| &\le \frac{c_1}{n^2} E_{n-3}(\partial_1 \partial_2 \partial_3) + \frac{c_2}{n^3} \mathcal{E}_{n-4}(f), \; i = 1,2,3, \\ \|\partial_i \partial_j f - \partial_i \partial_j p_n\| &\le \frac{c_1}{n} E_{n-3}(\partial_1 \partial_2 \partial_3) + \frac{c_2}{n^2} \mathcal{E}_{n-4}(f), \; 1 \le i,j \le 3. \end{aligned}$$

Xu, Constr. Approx. [2017]

Thank You

Happy 60, Thangavelu