HARDY AND FRACTIONAL HARDY INEQUALITY FOR DUNKL LAPLACIAN

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Theorem (Rupert L. Frank, 2011)

Let w be a positive function satisfying $-\Delta w + Vw \geq 0$ in $\mathbb{R}^N$ for some function V. Then for all $u \in C_0^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V|u|^2)dx \geq \int_{\mathbb{R}^N} |\nabla(w^{-1}u)|^2 w^2 dx.$$

Equality holds when $-\Delta w + Vw = 0$.

The theorem implies that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq -\int_{\mathbb{R}^N} V|u|^2 dx.$$

Choose $w(x) = |x|^{-(N-2)/2}$ and $V = -((N-2)/2)^2|x|^{-2}$. Then $-\Delta w + Vw = 0$. Substituting V and w in the above theorem we get following Hardy inequality.

Theorem

Let $N \geq 3$. Then for all $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx.$$

The inequality is strict for every $u \not\equiv 0$, but the constant $((N-2)/2)^2$ cannot be replaced by a smaller constant.

Similarly one can put $w(x) = e^{-\alpha|x|^2/2}$ and $V(x) = \alpha^2|x|^2 - \alpha N$ in the theorem and optimizing for α we get the Heisenberg uncertainty principle

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^N} |x|^2 |u|^2 dx\right)^{1/2} \geq \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 dx.$$

Let

$$\mathbb{R}^N_{k_+} = \{(x_1, ..., x_N) \in \mathbb{R}^N : x_{N-k+1} > 0, ..., x_N > 0\}.$$

Theorem (Dan Su, Qiao-Hua Yang - 2012)

Let $N \geq 3$. There holds, for all $u \in C_0^\infty(\mathbb{R}^N_{k_+})$,

$$\int_{\mathbb{R}^N_{k_+}} |\nabla u|^2 dx \geq \frac{(N-2+2k)^2}{4} \int_{\mathbb{R}^N_{k_+}} \frac{|u|^2}{|x|^2} dx,$$

and the constant $\frac{(N-2+2k)^2}{4}$ is sharp.

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote σ_α as the reflection in the hyper plane $\langle\alpha\rangle^\perp$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle\alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $|\alpha| := \sqrt{\langle\alpha, \alpha\rangle}$.

Definition

Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a finite set. Then R is called a root system, if
(1) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$
(2) $\sigma_\alpha(R) = R$ for all $\alpha \in R$.

Preliminaries of Dunkl Operator

- The group $G = G(R)$ which is generated by reflections $\{\sigma_\alpha : \alpha \in R\}$ is called reflection group (or Coxeter-group) associated with R.
- A $G-$ invariant function from the root system R to $\mathbb{R}_+$ is called a multiplicity function.
- $R = R_+ \sqcup (-R_+)$ where R_+ and $-R_+$ is separated by a hyperplane through the origin. Such a set R_+ is called positive subsystem.

Preliminaries of Dunkl Operator

- Let ∂_j denotes the partial derivative corresponding to e_j, and R is a fixed root system.

Definition

Let k be a multiplicative function. Then for $e_j \in \mathbb{R}^N$, the Dunkl operator $T_j := T_{e_j}(k)$ is defined (for $f \in C^1(\mathbb{R}^N)$) by

$$T_j f(x) := \partial_j f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, e_j \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

- For fixed k, the associated T_j commutes.

Preliminaries of Dunkl Operator

- Associated with the reflection group and the function k, the weight function h_k is defined by

$$h_k(x) = \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{k_\nu}, \quad x \in \mathbb{R}^N.$$

- This is positive homogeneous function of degree $\gamma_k := \sum_{\nu \in R_+} k_\nu = \sum_{\nu \in R_+} k(\nu)$ and is invariant under the reflection group G. Let us denote $\lambda_k = \frac{N-2}{2} + \gamma_k$.

Theorem

Let $k \geq 0$. Then for every $f \in \mathscr{S}(\mathbb{R}^N)$ and $g \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} T_j f(x) g(x) h_k^2(x) dx = -\int_{\mathbb{R}^N} f(x) T_j g(x) h_k^2(x) dx.$$

Preliminaries of Dunkl Operator

- If $f, g \in C^1(\mathbb{R}^N)$ and at least one of them is G-invariant then $T_j(fg) = T_j f.g + f.T_j g$.
- The Dunkl kernel $E_k(., y)$ is defined as the unique solution of the system,
$$T_j f = y_j f, \quad f(0) = 1.$$
- $E_k(x, y)$ is the analogous function of $e^{\langle x,y \rangle}$ as $e^{\langle .,y \rangle}$ is the solution of $\partial_j f = y_j f$ with $f(0) = 1$.
- Dunkl gradient: $\nabla_k = (T_1, T_2, \cdots, T_N)$
- Dunkl Laplacian : $\Delta_k = \sum_{j=1}^{N} T_j^2$

We will prove Hardy inequality in Dunkl setting and will deduce certain related results.

Theorem

Let w be positive radial function satisfying $-\Delta_k w + Vw \geq 0$ in $\mathbb{R}^N$ for some function V. Then for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (|\nabla_k u|^2 + V|u|^2) h_k^2(x) dx \geq \int_{\mathbb{R}^N} |\nabla_k (w^{-1} u)|^2 w^2 h_k^2(x) dx.$$

Equality holds when $-\Delta_k w + Vw = 0$.

Sketch of the proof

Let $u = wv$ and w is radial. We can assume u is real valued. Then

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 = \int_{\mathbb{R}^N} |w\nabla_k v + v\nabla_k w|^2$$

$$= \int_{\mathbb{R}^N} \left(|\nabla_k v|^2 w^2 + v^2 \, |\nabla_k w|^2 + 2wv \sum_{j=1}^{N} T_j v T_j w \right) h_k^2(x) dx$$

and estimate integral of each term separately.

Sketch of the proof

Substitute simplify and use the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |w\nabla_k v + v\nabla_k w|^2 h_k^2(x)dx \\
&\geq \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 h_k^2(x)dx - \int_{\mathbb{R}^N} wv^2\Delta_0 w h_k^2(x)dx \\
&-2\gamma_k \int_{\mathbb{R}^N} \frac{w(r)w'(r)}{r} v^2(x)h_k^2(x)dx \\
&= \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 h_k^2(x)dx - \int_{\mathbb{R}^N} v^2 w\Delta_k w h_k^2(x)dx
\end{aligned}
$$

So we can conclude as
$\int_{\mathbb{R}^N} |\nabla_k u|^2 h_k^2(x)dx \geq \int_{\mathbb{R}^N} |\nabla_k v|^2 w^2 h_k^2(x)dx - \int_{\mathbb{R}^N} Vu^2 h_k^2(x)dx$

Choose

$$w(x) = |x|^{-\lambda_k}, \quad \lambda_k = \frac{N-2}{2} + \gamma_k.$$

Using the Dunkl Laplacian for the radial function

$$\Delta_k = \frac{d^2}{dr^2} + \frac{2\lambda_k + 1}{r}\frac{d}{dr},$$

and equating $\Delta_k w = Vw$ we get

$$V(x) = -\lambda_k^2 |x|^{-2}.$$

Now use the above theorem and obtain

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 h_k^2(x)dx \geq \lambda_k^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} h_k^2(x)dx. \quad (1)$$

Let $\tilde{\nabla}_k = (\nabla_k, \frac{\partial}{\partial x_{N+1}})$ be the gradient on $\mathbb{R}^{N+1}$, where ∇_k is the Dunkl gradient on $\mathbb{R}^N$.

Lemma

For $l \in \{1/2, 1, 3/2, 2, ..., N/2, ..\}$ and for $u \in C_0^\infty(\mathbb{R}_+^N)$,

$$\int_{\mathbb{R}_+^{N+1}} |\tilde{\nabla}_k u|^2 h_k^2(x) dx dx_{N+1} + l(l-1) \int_{\mathbb{R}_+^{N+1}} \frac{|u(x)|^2}{x_{N+1}^2} h_k^2(x) dx dx_{N+1}$$
$$\geq \frac{(N+2\gamma_k+2l-1)^2}{4} \int_{\mathbb{R}_+^{N+1}} \frac{|u(x)|^2}{|x|^2} h_k^2(x) dx dx_{N+1},$$

where $\frac{(N+2\gamma_k+2l-1)^2}{4}$ is sharp.

Sketch of the proof

The proof mainly rely on the relation

- $$x_{N+1}^{-l}\left(\sum_{j=1}^{N} T_j^2 + \frac{\partial^2}{\partial x_{N+1}^2} + \frac{l(l-1)}{x_{N+1}^2}\right)x_{N+1}^{l} g(x)$$
  $$= \left(\sum_{j=1}^{N} T_j^2 + \frac{\partial^2}{\partial x_{N+1}^2} + \frac{2l}{x_{N+1}}\frac{\partial}{\partial x_{N+1}}\right) g(x)$$

  AND
- The Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 h_k^2(x)dx \geq \lambda_k^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} h_k^2(x)dx.$$

Hardy inequality for cone

Let $\tilde{\nabla}_l = (\nabla_k, \frac{\partial}{\partial x_{N-l+1}}, ..., \frac{\partial}{\partial x_N})$, where ∇_k is the Dunkl gradient on $\mathbb{R}^{N-l}$.

Theorem

Let $N + 2\gamma \geq 3$ and $u \in C_0^\infty(\mathbb{R}^N_{l_+})$. Then the following inequality holds:

$$\int_{\mathbb{R}^N_{l_+}} |\tilde{\nabla}_k u|^2 h_k^2(x) dx dx_{N-l+1}..dx_N$$
$$\geq \frac{(N+2l+2\gamma_k-2)^2}{4} \int_{\mathbb{R}^N_{l_+}} \frac{|u|^2}{|x|^2} h_k^2(x) dx dx_{N-l+1}..dx_N,$$

where the constant $\frac{(N+2l+2\gamma_k-2)^2}{4}$ is sharp.

Motivated by the numerical identity

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}, \lambda > 0,$$

we can define the fractional power of the Dunkl Laplacian denoted by Δ_k^s.

Definition

Bochner's definition: For $s \in (0,1)$, the fractional Dunkl Laplacian Δ_k^s is defined by

$$\Delta_k^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta_k} f(x) - f(x)) \frac{dt}{t^{1+s}}.$$

Theorem

Let $0 < s < 1$ *and* $f \in C_0^\infty(\mathbb{R}_+^N)$. *Then*

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho$$
$$\geq c_h \frac{\Gamma(1-\frac{s}{2}-\gamma_k)}{\Gamma(\frac{s}{2}-\gamma_k)} \frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} \int_{\mathbb{R}^N} \frac{|u(x,0)|^2}{(1+|x|^2)^s} h_k^2(x) dx,$$

where $c_h = \prod_{\nu \in R_+} \rho^{2k(\nu)}$.

Sketch of the proof

Assume u is real valued. Choose v is radial and using Cauchy-Schwarz inequality.

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u - \frac{u}{v}\nabla_k v|^2 \rho^a h_k^2(x) dx d\rho$$
$$\leq \int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u|^2 \rho^a h_k^2(x) dx d\rho + \int_0^\infty \int_{\mathbb{R}^N} \frac{u^2}{v} \rho^a L_a^k v h_k^2(x) dx d\rho$$
$$+ \int_{\mathbb{R}^N} \frac{u^2(x,0)}{v(x,0)} \lim_{\rho\to 0} (\rho^a \frac{\partial v}{\partial \rho})(x,\rho) h_k^2(x) dx$$

where L_a^k is the differential operator

$$L_a^k = \Delta_k + \partial_\rho^2 + \frac{a}{\rho}\partial_\rho,$$

and

$$\Delta_k = \sum_{j=1}^N T_j^2.$$

Sketch of the proof

- $L_a^k = \Delta_k + \partial_\rho^2 + \frac{a}{\rho}\partial_\rho$.
- Look for v such that $L_a^k v = 0$ with initial condition $v(x,0) = f(x)$.
-

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u(x,\rho)|^2 \rho^a h_k^2(x) dx d\rho$$
$$\geq - \int_{\mathbb{R}^N} \frac{u^2(x,0)}{v(x,0)} \lim_{\rho\to 0} (\rho^a \frac{\partial v}{\partial \rho})(x,\rho) h_k^2(x) dx$$

- $a = M - 1$ is a positive integer, L_a^k is given by the action of $\tilde{\Delta}_k = \Delta_k + \Delta_o$ on $\mathbb{R}^{N+M}$ on functions $v(x,y)$ which are radial in the y variable.

Sketch of the proof

- $$v(x,\rho) = (\rho^2 + |x|^2)^{-\frac{N+M}{2} - \gamma + 1}$$

  solves $L_a^k v = 0$ even if a is not a positive integer.
- Define ψ_α, for any $\alpha \geq 0$, $\psi_\alpha(x) = (1 + |x|^2)^{-\alpha}$
- Write
  $$v(x,\rho) = \rho^{-(N-s)-2\gamma_k} \psi_{\frac{N+s}{2}+\gamma_k}(\rho^{-1}x)$$

Denote

$$v_{s,\rho} = \rho^s(\rho^2 + |x|^2)^{-\frac{N+s}{2}-\gamma_k} = \rho^{-N-2\gamma_k}\psi_{\frac{N+s}{2}+\gamma_k}(\rho^{-1}x).$$

Also $f * v_{s,\rho}$ satisfies

$$(\Delta_k + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)u = 0.$$

For $s \leq N/2$ we have $\lim_{\rho\to 0} f *_k v_{s,\rho} = a_N(s)f$, where

$$a_N(s) = \int_{\mathbb{R}^N}(1+|x|^2)^{-\frac{N+s}{2}-\gamma_k}dx$$

Now

$$\lim_{\rho\to 0} -\frac{\rho^{1-s}\partial_\rho(f *_k v_{s,\rho})}{f *_k v_{s,\rho}} = \frac{\Gamma(1-\frac{s}{2}-\gamma_k)}{\frac{s}{2}-\gamma_k}\frac{(\Delta_k^{s/2} f)}{f}$$

Recall that,

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u(x,\rho)|^2 \rho^a h_k^2(x)dxd\rho$$
$$\geq -\int_{\mathbb{R}^N} \frac{u^2(x,0)}{v(x,0)} \lim_{\rho\to 0}(\rho^a \frac{\partial v}{\partial \rho})(x,\rho)h_k^2(x)dx$$

Now we get the following inequality:

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho$$
$$\geq c_h \frac{\Gamma(1-\frac{s}{2}-\gamma_k)}{\Gamma(\frac{s}{2}-\gamma_k)} \int_{\mathbb{R}^N} \frac{u^2(x,0)}{f(x)} (\Delta_k^{s/2} f)(x) h_k^2(x) dx,$$

where $c_h = \prod_{\nu \in R_+} \rho^{2k(\nu)}$.

Choose $f(x) = \psi_{\frac{N-s}{2}+\gamma_k}(x) = (1+|x|^2)^{-\frac{N-s}{2}-\gamma_k}$. Use

$$\Delta_k^{s/2} \psi_{\frac{N-s}{2}+\gamma_k}(x) = \frac{\Gamma(\frac{N+s}{2}+\gamma_k)}{\Gamma(\frac{N-s}{2}+\gamma_k)} \psi_{\frac{N+s}{2}+\gamma_k}$$

Trace Hardy Inequality for Dunkl fractional Laplacian for homogeneous weight

Theorem

For $0 < s < 1$ *and* $f \in C_0^\infty(\mathbb{R}^N_+)$*, then*

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla_k u(x,\rho)|^2 \rho^{1-s} h_k^2(x) dx d\rho$$
$$\geq c_h w_N^k \frac{(-i)^{-N}}{2^{\gamma_k + \frac{N}{2} + 1}} \frac{\Gamma(1 - \frac{s}{2} - \gamma_k)\Gamma(\frac{N}{2} + 2\gamma_k)}{\Gamma(\frac{N-s}{2} + \gamma_k)} \int_{\mathbb{R}^N} \frac{u^2(x,0)}{|x|^s} h_k^2(x) dx$$

Take $f = \varphi *_k u_{-s,\delta}(x)$ where $u_{-s,\delta}(x) = (\delta^2 + |x|^2)^{-\frac{n-s}{2} - \gamma_k}$ and proceed as in the last proof.

The following Hardy inequality can be obtained as a corollary of the Trace Hardy inequality.

Theorem

Let $f \in L^2(\mathbb{R}^N, h_k^2(x))$ *be such that* $\Delta_k^s f \in L^2(\mathbb{R}^N, h_k^2(x))$*. For* $0 < s < 1$*, one has*

$$\langle \Delta_k^s f, f \rangle \geq \frac{\Gamma(\frac{N+s}{2} + \gamma_k)}{\Gamma(\frac{N-s}{2} + \gamma_k)} \int_{\mathbb{R}^N} \frac{f(x)^2}{(1+|x|^2)^s} h_k^2(x) dx.$$

The non-homogeneous form of the fractional Hardy inequality for Dunkl Laplacian is obtained from the Trace Hardy inequality with non-homogeneous weight.

Hardy Inequality for the Dunkl fractional Laplacian with homogeneous weight

Theorem

Let $N \geq 1$ and $0 < s < 1$ be such that $N/2 > s$. Then for $f \in C_0^\infty(\mathbb{R}^N)$, we have

$$E_{N,s} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^{2s}} h_k^2(x)dx \leq \langle \Delta_k^s f, f \rangle,$$

where the constant $E_{N,s}$ is given by

$$E_{N,s} = 4^s \left(\frac{\Gamma(\frac{N}{4} + \frac{\gamma_k}{2} + \frac{s}{2})}{\Gamma(\frac{N}{4} + \frac{\gamma_k}{2} - \frac{s}{2})} \right)^2.$$

Remark:

- Hardy inequality related to Dunkl gradient for upper half plane and cone was proved using Hardy in equality on $\mathbb{R}^N$ for Dunkl gradient
- Fractional Hardy inequality related to Dunkl Laplacian for half-space and cone can be obtained using fractional Hardy inequality for Dunkl Laplacian.

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THANK YOU!

Hardy Inequality for the Dunkl fractional Laplacian for half-space

Theorem

Let $u \in C_0^\infty(\mathbb{R}^N_+)$. *For* $0 < s < 1$ *and* $N \geq 1$, *one has*

$$\langle \tilde{\Delta}^s_{k_2} u, u \rangle_{\mathbb{R}^N_+} \geq \frac{\Gamma(\frac{N+2+2s}{2} + \gamma)}{\Gamma(\frac{N+2-2s}{2} + \gamma)} \int_{\mathbb{R}^N_+} \frac{u(x, x_N)^2}{(1 + |x|^2 + x_N^2)^{2s}} h_k^2(x) dx dx_N.$$

Theorem

Let $0 < s < 1$ *and* $N > s$. *For* $u \in C_0^\infty(\mathbb{R}^N_{k_+})$, *the following inequality holds:*

$$\langle \tilde{\Delta}^s_{k_2} u, u \rangle_{\mathbb{R}^N_{k_+}} \geq \frac{\Gamma(\frac{N+2k+2s}{2} + \gamma)}{\Gamma(\frac{N+2k-2s}{2} + \gamma)} \int_{\mathbb{R}^N_{k_+}} \frac{u^2}{(1 + |x|^2 + x_n^2)^{2s}} h_k^2(x) dx dx_N.$$

Hardy Inequality for the Dunkl fractional Laplacian with homogeneous weight

Theorem

Let $u \in C_0^\infty(\mathbb{R}^N)$ and $n/2 \geq s$. Then for $0 < s < 1$ we have

$$\langle \tilde{\Delta}_{k_2}^s u, u \rangle_{\mathbb{R}_+^N} \geq 4^s \left(\frac{\Gamma(\frac{N+2}{4} + \frac{\gamma}{2} + \frac{s}{2})}{\Gamma(\frac{N+2}{4} + \frac{\gamma}{2} - \frac{s}{2})} \right)^2 \int_{\mathbb{R}_+^N} \frac{|u(x, x_N)|^2}{|x|^{2s}} h_k^2(x) dx dx_N.$$

Hardy Inequality for the Dunkl fractional Laplacian for cone with homogeneous weight

Theorem

Let $u \in C_0^\infty(\mathbb{R}^N)$ *and* $N/2 \geq s$. *Then for* $0 < s < 1$ *we have*

$$\langle \tilde{\Delta}_{k_2}^s u, u \rangle_{\mathbb{R}^N_{k_+}} \geq 4^s \left(\frac{\Gamma(\frac{N+2k}{4} + \frac{\gamma}{2} + \frac{s}{2})}{\Gamma(\frac{N+2k}{4} + \frac{\gamma}{2} - \frac{s}{2})} \right)^2 \int_{\mathbb{R}^N_{k_+}} \frac{|u(x, x_N)|^2}{|x|^{2s}} h_k^2(x) dx dx_N.$$