Sparse bounds and sharp weighted inequalities

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Sparse families and sparse bounds

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  3. Fix a cube $Q_0 \subset \mathbb{R}^n$. Take an arbitrary collection of pairwise disjoint cubes $Q_1^j \subset Q_0$ such that $\sum_j |Q_1^j| \leqslant (1-\eta)|Q_0|$. In a similar way take collections of cubes in every Q_1^j, and so on. Then the resulting family of all the cubes appearing in the process will be η-sparse.

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$$M^{\mathcal{D}}f(x) = \sup_{Q\ni x, Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q |f|,$$

and consider the sets $\Omega_k = \{x : M^{\mathcal{D}}f(x) > 2^{(n+1)k}\}, k \in \mathbb{Z}$.

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- By a sparse bound (or a sparse domination) for a given operator T one typically means an estimate of the form

$$|\langle Tf, g\rangle| \leqslant C \sum_{Q\in\mathcal{S}} \langle f\rangle_{p,Q}\langle g\rangle_{r,Q}|Q| \quad (1 \leqslant p, r < \infty),$$

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where $\mathcal{S}$ is a sparse family (depending on f and g but with an absolute sparseness constant).
- It has been observed that such an estimate with the best possible (that is, the smallest possible) p and r typically yields the sharp quantitative weighted inequalities for T.

Some basic operators

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  2. T is represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \quad \text{for all } x \notin \operatorname{supp} f;$$

  3. K satisfies the size condition $|K(x,y)| \leqslant \frac{C_K}{|x-y|^n}, x \neq y$;
  4. K satisfies the regularity condition

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leqslant \omega\left(\frac{|x-x'|}{|x-y|}\right) \frac{1}{|x-y|^n}$$

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- The standard assumption on ω is that $\omega(t) = Ct^\delta, 0 < \delta \leqslant 1$. In this case we will skip ω. More general assumptions are

$$\int_0^1 \omega(t) \log\frac{1}{t}\frac{dt}{t} < \infty \quad (\log -\text{Dini}), \qquad \int_0^1 \omega(t)\frac{dt}{t} < \infty \quad (\text{Dini}).$$

Sharp quantitative weighted inequalities

- By a weight we mean a non-negative, locally integrable function. Given a weight w, set

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- S. Buckley (1993): for the maximal operator,
$$\|M\|_{L^p(w)\to L^p(w)} \leqslant C(n,p)[w]_{A_p}^{\frac{1}{p-1}} \quad (p > 1),$$
and the exponent $\frac{1}{p-1}$ is best possible for every $p > 1$.

The A_2 conjecture

- For Calderón-Zygmund operators T, if α is the best possible exponent in
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- The key idea of the proof: a representation of Calderón-Zygmund operators in terms of the so-called Haar shift operators
$$\mathbb{S}_{\mathscr{D}}^{m,k} f(x) = \sum_{Q \in \mathscr{D}} \sum_{\substack{Q', Q'' \in \mathscr{D}, Q', Q'' \subset Q \\ \ell(Q') = 2^{-m} \ell(Q), \ell(Q'') = 2^{-k} \ell(Q)}} \frac{\langle f, h_{Q'}^{Q''} \rangle}{|Q|} h_{Q''}^{Q'}(x)$$
with their subsequent analysis.

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A.L. (2009), T.Hytönen (2012), A.L. and F. Nazarov (2014)

For every measurable function f with $\mu_f(\alpha) < \infty$, there exists a $\frac{1}{6}$-sparse family $\mathcal{S} \subset \mathscr{D}$ such that for a.e. $x \in \mathbb{R}^n$,

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A sparse domination approach to the A_2 conjecture

- The oscillation of f on E:

$$\omega(f;E) = \sup_E f - \inf_E f.$$

Given $0 < \lambda < 1$, define the λ-oscillation of f over a cube Q by

$$\omega_\lambda(f;Q) = \inf\{\omega(f;E) : E \subset Q, |E| \geqslant (1-\lambda)|Q|\}.$$

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- Some history:
  1. L. Carleson (1976): a different proof of the $H^1 - BMO$ duality;
  2. J. Garnett and P. Jones (1982): a dyadic version;
  3. N. Fujii (1991): BMO can be replaced by L^1_{loc}.

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- A.L. (2012): for every Calderón-Zygmund operator T,

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  where the supremum is taken over all $\frac{1}{2}$-sparse families $\mathcal{S}\subset\mathscr{D}$ and all dyadic lattices $\mathscr{D}$.

The proof of $\|A_{\mathcal{S},\mathscr{D}}\|_{L^2(w)\to L^2(w)} \leqslant C[w]_{A_2}$.

Let $A_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} f_Q \chi_Q(x)$, where $\mathcal{S} \subset \mathscr{D}$ and $\mathcal{S}$ is $\frac{1}{2}$-sparse. Denote

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- (2016 –): $\approx$ 60 "sparse domination" papers.

The maximal operator M_T

Given a sublinear operator T, define the maximal operator M_T by

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- Assume now that T is an ω-CZ operator, and define the maximal truncated operator by

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- By $(*)$, M_T is of weak type $(1,1)$ and

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Theorem (A.L. (2015))

Assume that T and M_T are of weak type $(1,1)$. Then, for every compactly supported $f \in L^1(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S}$ such that for a.e. x,

$$|Tf(x)| \leqslant K A_{\mathcal{S}} |f|(x),$$

where $K = C_n(\|T\|_{L^1 \to L^{1,\infty}} + \|M_T\|_{L^1 \to L^{1,\infty}})$.

The proof of $|Tf(x)| \leqslant KA_{\mathcal{S}}|f|(x)$

- The key recursive claim: there exist pairwise disjoint cubes $P_j \subset Q_0$ such that $\sum_j |P_j| \leqslant \frac{1}{2}|Q_0|$ and for a.e. on Q_0,
$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leqslant K|f|_{3Q_0} + \sum_j |T(f\chi_{3P_j})|\chi_{P_j}.$$

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$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leqslant K \sum_{Q\in\mathcal{F}} |f|_{3Q}\chi_Q(x).$$
- Take a partition of $\mathbb{R}^n$ by cubes R_j such that $\operatorname{supp}(f) \subset 3R_j$ for each j, and apply the above estimate to each R_j instead of Q_0. Let $\mathcal{F}_j$ be the corresponding sparse family of the cubes from R_j. Setting $\mathcal{F} = \cup_j \mathcal{F}_j$, we obtain that $\mathcal{F}$ is $\frac{1}{2}$-sparse and for a.e. $x \in \mathbb{R}^n$,
$$|Tf(x)| \leqslant K \sum_{Q\in\mathcal{F}} |f|_{3Q}\chi_Q(x).$$
- Hence, the statement holds with the $\frac{1}{2\cdot 3^n}$-sparse family
$$\mathcal{S} = \{3Q : Q \in \mathcal{F}\}.$$

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- The key recursive claim: there exist pairwise disjoint cubes $P_j \subset Q_0$ such that $\sum_j |P_j| \leqslant \frac{1}{2}|Q_0|$ and for a.e. on Q_0,

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$$\begin{aligned} |T(f\chi_{3Q_0})|\chi_{Q_0} &\leqslant |T(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j} \\ &+ \sum_j |T(f\chi_{3P_j})|\chi_{P_j}. \end{aligned}$$

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- Take C_n such that $|E = A \cup B| \leqslant \frac{1}{2^{n+2}}|Q_0|$, where
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- From this, 1, 2 and 3 follow.

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- The "classical" dyadic lattice

$$\{2^{-k}([0,1)^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n\}$$

is not a dyadic lattice in this sense.

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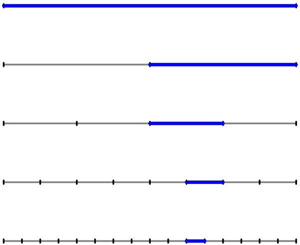
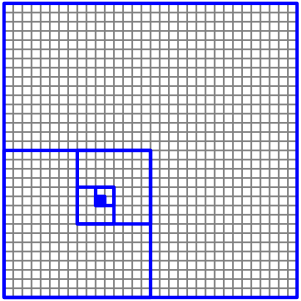
  - for every compact set $K \subset \mathbb{R}^n$, there is a cube $Q \in \mathscr{D}$ containing K.
- In order to construct a dyadic lattice $\mathscr{D}$, it suffices to fix any cube Q_0 and then expand it dyadically, including all dyadic children into $\mathscr{D}$.





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For every dyadic lattice $\mathscr{D}$, there exist 3^n dyadic lattices $\mathscr{D}^{(1)}, \dots, \mathscr{D}^{(3^n)}$ such that

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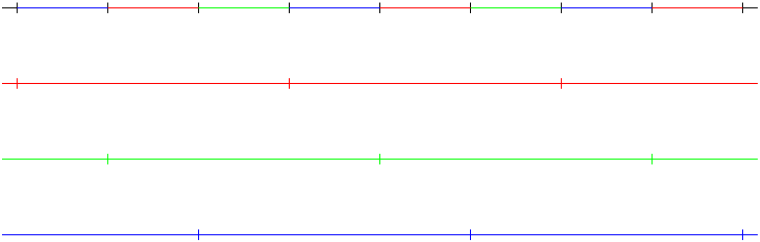

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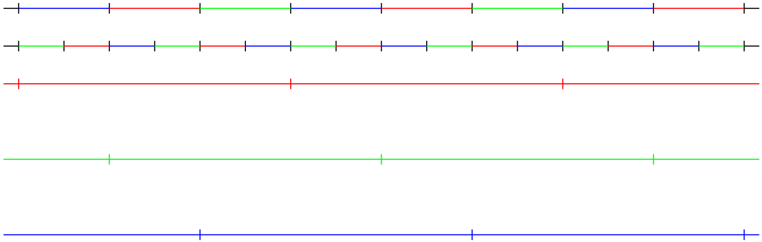

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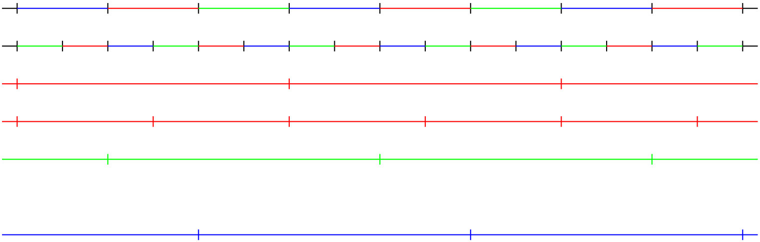

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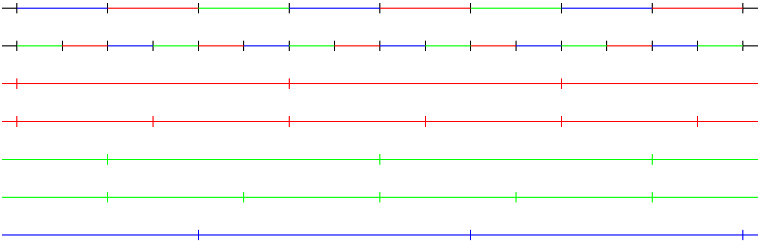

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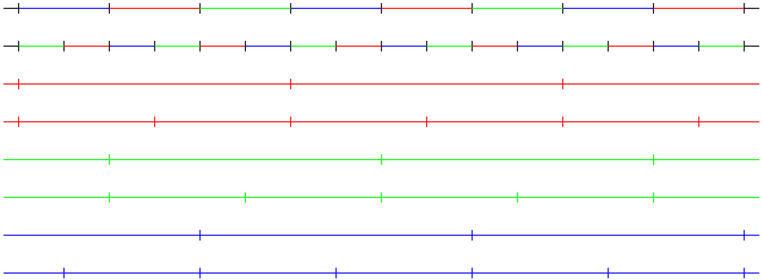

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- Proof: fix a dyadic lattice $\mathscr{D}$. Let $Q \subset \mathbb{R}^n$. Take a cube $Q' \in \mathscr{D}$ containing the center of Q and such that $\ell_Q/2 < \ell_{Q'} \leqslant \ell_Q$. Then $Q \subset 3Q'$. But $3Q' \in \mathscr{D}^{(j)}$.

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$$\sum_{Q \in \mathcal{S}} |f|_Q \chi_Q \leqslant 3^n \sum_{Q \in \mathcal{S}} |f|_{P_Q} \chi_{P_Q} \leqslant 3^n \sum_{j=1}^{3^n} \sum_{P \in \mathcal{S}_j} |f|_P \chi_P.$$

A general sparse domination principle

- We have seen that if T and
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Let $r, s \geqslant 1$. Assume that T is a sublinear operator of weak type (q,q) for some $1 \leqslant q \leqslant r$, and $\mathcal{M}_T$ maps $L^r \times L^s$ into $L^{\nu,\infty}$, where $\frac{1}{\nu} = \frac{1}{r} + \frac{1}{s}$. Then, for every compactly supported $f \in L^r(\mathbb{R}^n)$ and every $g \in L^s_{loc}$, there exists a $\frac{1}{2\cdot 3^n}$-sparse family $\mathcal{S}$ such that $(*)$ holds, where

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Corollary

If T is of weak type (q,q) for some $1 \leqslant q \leqslant r$ and $M_{s',T}$ is of weak type (r,r), then for every compactly supported $f \in L^r(\mathbb{R}^n)$ and every $g \in L^s_{loc}$, there exists a $\frac{1}{2\cdot 3^n}$-sparse family $\mathcal{S}$ such that

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where $K = C_n(\|T\|_{L^q \to L^{q,\infty}} + \|M_{s',T}\|_{L^r \to L^{r,\infty}})$.

A general sparse domination principle

- Given $1 \leqslant p \leqslant \infty$, define
$$M_{p,T}f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |T(f\chi_{\mathbb{R}^n \setminus 3Q})|^p dy \right)^{1/p}$$
- By Hölder's inequalities,
$$\|\mathcal{M}_T\|_{L^r \times L^s \to L^{\nu,\infty}} \leqslant C_n \|M_{s',T}\|_{L^r \to L^{r,\infty}} \quad (1/\nu = 1/r + 1/s).$$

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where $K = C_n(\|T\|_{L^q \to L^{q,\infty}} + \|M_{s',T}\|_{L^r \to L^{r,\infty}})$.

- If $s = 1$, then a stronger, pointwise estimate holds:
$$|Tf(x)| \leqslant K \sum_{Q \in \mathcal{S}} \langle f \rangle_{r,Q} \chi_Q(x).$$

Rough singular integrals

- Consider a class of rough homogeneous singular integrals defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(y/|y|)}{|y|^n} dy,$$

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- A.L. (2017): For T_Ω with $\Omega\in L^\infty$,
$$\|M_{p,T_\Omega}f\|_{L^{1,\infty}} \leqslant C_n\|\Omega\|_{L^\infty}p\|f\|_{L^1} \quad (p>1).$$

A reformulation in terms of $M_{\lambda,T}$

$$\boxed{\|M_{p,T_\Omega} f\|_{L^{1,\infty}} \leqslant C_n \|\Omega\|_{L^\infty} p \|f\|_{L^1} \quad (*)}$$

- Consider

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- $(*)$ is equivalent to
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Some words about the proof

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- The key statement: if $\|T_\Omega\|_{L^2\to L^2} \lesssim \delta$, then

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  This part is based heavily on the decomposition of A. Seeger (1996).

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- Applying the key statement with $\Omega = \Omega - \Omega_\varepsilon$ and $\delta = \varepsilon^{1/2}$ yields

$$\|M_{\lambda,T_\Omega}\|_{L^1 \to L^{1,\infty}} \lesssim \Big(\frac{\varepsilon^{1/2}}{\lambda} + \log \frac{\mathrm{e}}{\varepsilon}\Big).$$

  It remains to optimize the argument with respect to ε: take $\varepsilon = \lambda^2$.

A sharp quadratic bound

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$$\|T_\Omega\|_{L^2(w)\to L^2(w)} \leqslant C_n \|\Omega\|_{L^\infty} [w]_{A_2}^2.$$

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$$\|M \circ T_\Omega\|_{L^2(w)\to L^2(w)} \leqslant C_n \|\Omega\|_{L^\infty}[w]_{A_2}^2, \tag{1}$$

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$$M(T_\Omega f)(x) \lesssim MMf(x) + M_{1,T_\Omega} f(x)$$
and
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- The second estimate along with $\|M_{p,T_\Omega}\|_{L^1\to L^{1,\infty}} \lesssim p$ implies that

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- Hence, $\|M_{1,T_\Omega}\|_{L^2(w)\to L^2(w)}\lesssim [w]_{A_2}^2$, and (1) follows from the first estimate along with Buckley's linear $[w]_{A_2}$ bound for M.

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- The second estimate along with $\|M_{p,T_\Omega}\|_{L^1\to L^{1,\infty}} \lesssim p$ implies that
$$\|M_{p,(M_{1,T_\Omega})}\|_{L^1\to L^{1,\infty}} \lesssim p.$$
- The sharpness of (1) follows from
$$\|M \circ T_\Omega\|_{L^p\to L^p} \geqslant \frac{c}{(p-1)^2}$$
as $p \to 1$, and a general extrapolation argument found by T. Luque, C. Pérez and E. Rela (2015).

An open question

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and the exponent $2\max\left(1,\frac{1}{p-1}\right)$ is best possible.

Two-weighted theory

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A sparse domination of commutators

- Introduce the sparse operator $\mathcal{T}_{\mathcal{S},b}$ defined by

$$\mathcal{T}_{\mathcal{S},b}f(x) = \sum_{Q\in\mathcal{S}} |b(x) - b_Q| f_Q \chi_Q(x).$$

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- A.L., S. Ombrosi, I. Rivera-Ríos (2016): for every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there are $\frac{1}{2\cdot 9^n}$-sparse families $\mathcal{S}_j \subset \mathscr{D}^{(j)}, j = 1, \dots, 3^n$, such that for a.e. $x \in \mathbb{R}^n$,
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  - If $\lambda = \mu = w$, this recovers the sharp bound by Chung-Pereyra-Pérez:

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- Consider $\mathcal{T}^{\star}_{\mathcal{S},b}f(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|} \int_Q |b - b_Q| f \right) \chi_Q(x).$

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- The key technical ingredient is the following: given a dyadic lattice $\mathscr{D}$ and a sparse family $\mathcal{S}\subset\mathscr{D}$, there exists a sparse family $\tilde{\mathcal{S}}\subset\mathscr{D}$ containing $\mathcal{S}$ and such that if $Q\in\tilde{\mathcal{S}}$, then for a.e. $x\in Q$,

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- Applying
$$\|A_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \leqslant c_{n,p}[w]_{A_p}^{\max\left(1,\frac{1}{p-1}\right)}$$
twice yields $(*)$.

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  - On the other hand, the assumption $b\in BMO_{\nu^{1/m}}$ appeared much earlier, in the work of J. García-Cuerva, E. Harboure, C. Segovia, J.L. Torrea (1991) about commutators of strongly singular integrals.

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$$T_\Omega f(x) = \int_{\mathbb{R}^n} \Omega\Big(\frac{x-y}{|x-y|}\Big)\frac{1}{|x-y|^n} f(y)dy \quad (x \notin \operatorname{supp} f),$$

where Ω is a measurable function on S^{n-1}, which does not change sign and is not equivalent to zero on some open subset from S^{n-1}. If there is $C > 0$ such that for every bounded measurable set $E \subset \mathbb{R}^n$,

$$\|(T_\Omega)_b^m(\chi_E)\|_{L^p(\lambda)} \leqslant C\mu(E)^{1/p},$$

then $b \in BMO_{\nu^{1/m}}$.

Thank you for your attention!