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Titchmarsh theorem for Fourier transform of Hölder-Lipschitz functions on compact homogeneous manifolds

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(Joint work with J. DELGADO and M. RUZHANSKY)

- Lipschitz([1864])
  Lipschitz condition states that

$$|f(x)-f(y)| \leq M|x-y|^{\alpha};\ 0<\alpha \leq 1.$$

He proved that this inequality is sufficient to have that the Fourier series of f converges everywhere to the value of f. If we denote

$$w(h,f)=\sup_{|x-y|<h} |f(x)-f(y)|$$

the modulus of continuity, Lipschitz condition can be written as :(Landau's notation)

$$w((h,f)=O(h^{\alpha}),\quad 0<\alpha \leq 1.$$

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- O.Szasz's theorem ([1922])
  Let $\mathbb{T}$ be the circle group, if

$$f \in \mathbb{L}ip_{\mathbb{T}}(\alpha, p) = \{f \in \mathbb{L}^p(\mathbb{T}) : \|\tau_h f - f\|_p = O(|h|^\alpha), \text{ as } h \to 0\},$$

then $\widehat{f} \in l^r$, where $\begin{cases} \alpha > \frac{1}{p} + \frac{1}{r} - 1, & \text{if } 1 < p \leq 2\,; \\ \alpha > \frac{1}{r} - \frac{1}{2}, & \text{if } p > 2. \end{cases}$

- E.C.Titchmarsh's theorem ([1927])

$$\mathbb{L}ip_{\mathbb{R}}(\alpha,p)=\{f\in\mathbb{L}^p(\mathbb{R}):\|\tau_h f-f\|_p=O(h^\alpha),\ \texttt{as}\ h\to 0\},$$

Theorem A ([Ti], Th 84) :

Let $0<\alpha\leq 1$ and $1<p\leq 2$.
If $f\in\mathbb{L}ip_{\mathbb{R}}(\alpha,p)$, then its Fourier transform $\widehat{f}$ belong to $\mathbb{L}^\beta$ for

$$\frac{p}{p+\alpha p-1}<\beta\leq\frac{p}{p-1}.$$

- Case $p = 2$

Theorem B([Ti], Th 85) :

$$f \in \mathbb{L}ip(\alpha, 2) \Leftrightarrow \int_{|x|\geq r} |\widehat{f}(x)|^2 dx = O(r^{-2\alpha}) \quad \texttt{as} \quad r \to \infty.$$

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- Younis [1970] $\mathbb{R}^n, \mathbb{T}^n$

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- Younis [1970] $\mathbb{R}^n, \mathbb{T}^n$
- Younis [1986]
  He showed that the result of Titchmarsh's theorem A does not holds for Dini-Lipschitz functions : It does not improve the Hausdorff-Young inequality and the conclusion is that $\widehat{f} \in \mathbb{L}^{p'}(\mathbb{R})$. Therefore, he considered some conditions which are rather situated in between the Lipschitz and Dini-Lipschitz conditions. These were inspired from Weiss and Zygmund [1959].

- Younis-Dini-Lipschitz conditions :

$$w(h, f) = O(h^{\alpha} ln(\frac{1}{h})^{-\delta}), \text{ where } \delta \geq 0.$$

He showed that Titchmarsh's theorem A and B could be extend to other setting as :

  - Compact group [1974]

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  - Daher-El ouadih [2017] (Th B) for Fourier Jacobi expansion.
  - .....etc

The aim of this talk is to extend the Titchmarsh's theorems to the setting of general compact homogeneous manifolds.

As an application of such extension, we derive a Fourier multiplier theorem for $\mathbb{L}^2$-Lipschitz spaces.

- Version of Titchmarsh's theorems on the $\mathbb{T}$ =Circle group

Theorem A :

Let $0 < \alpha \leq 1$ and $1 < p \leq 2$.
If $f \in \mathbb{L}ip_{\mathbb{T}}(\alpha, p)$, then its Fourier transform $\widehat{f}$ belongs to $\mathbb{L}^{\beta}(\mathbb{Z})$ for

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⋆ The lower bound $\frac{p}{p+\alpha p-1}$ is sharp can be proved by means of Hardy and Littlwood's function :

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{in\log n}}{n^{\frac{1}{2}+\alpha}} e^{inx},\ 0 < \alpha \leq 1$$

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⋆ $f \in \mathbb{L}ip_{\mathbb{T}}(\alpha, 2)$ BUT $\widehat{f} \notin l^{\frac{2}{2\alpha}}(\mathbb{Z})$(see[Z]).

Theorem B : p=2

Let $0 < \alpha \leq 1$; $f \in \mathbb{L}^2(\mathbb{T}^1)$, then

$$f \in \mathbb{L}ip_{\mathbb{T}^1}(\alpha, 2) \Leftrightarrow \sum_{|j| \geq N} |\widehat{f}(j)|^2 = O(N^{-2\alpha}) \text{ as } N \to \infty.$$

- Younis-Titchmarsh's theorem C :
  Similar theorems, we have only to replace $O(h^\alpha)$ by Y.D.Lipschitz condition$O(h^\alpha(\log(\frac{1}{|h|}))^\delta)$ as $h \to 0$.

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- If ξ is matrix representation, we have $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ where d_ξ is the dimension of the representation space of ξ.
- For $[\xi] \in \widehat{G}$, we can view ξ as a matrix-value function

$$\xi : G \longrightarrow \mathbb{C}^{d_\xi \times d_\xi}$$

- Fourier inversion formula :

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi Tr(\xi(x)\widehat{f}(\xi)).$$

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- For each $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions for the Laplacian $\mathcal{L}_G$ with the same eigenvalue which we denote by $-\lambda^2_{[\xi]}$ so that

$$-\mathcal{L}_G\xi_{ij}(x) = \lambda^2_{[\xi]}\xi_{ij}(x)$$

for all $1 \leq i, j \leq d_\xi$.

- Parseval identiy :

$$\|f\|_{\mathbb{L}^2(G)} = \left(\sum_{|\xi| \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|^2_{HS} \right)^{1/2}$$

where $\|\widehat{f}(\xi)\|^2_{HS} = Tr(\widehat{f}(\xi))\widehat{f}(\xi)^*)$. which gives the norm on $l^2(\widehat{G})$.

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- The weight for measuring the decay or growth of Fourier coefficients in this setting is

$$\langle \xi \rangle = (1 + \lambda_{[\xi]}^2)^{1/2}.$$

the eigenvalues of the elliptic first-order pseudo-differential operator $(I - \mathcal{L}_G)^{1/2}$.

- Taylor expansion :
  for $f \in C^{\infty}(G)$.

$$f(x) = \sum_{|\alpha| \leq N-1} D^N f(e) q_\alpha(x) + O(|x|^N)$$

- ⋆ for some invariant differential operators $D^{(\alpha)}$ of order $|\alpha|$.
  for an admissible family of function q_α.
- ⋆ $|x|$ denoting the geodesic distance from x to e.

Definition :

Let G be a compact Lie group. Let $0 < \alpha \leq 1$ and $1 \leq p\infty$.
We define the space $\mathbb{L}ip_G(\alpha, p)$.
$\mathbb{L}ip_G(\alpha, p) = \{f \in \mathbb{L}^p(G), \|f(h.) - f(.)\|_{\mathbb{L}^p(G)} = O(|h|^\alpha)$ as $|h| \to 0\}$
for $1 \leq p < \infty$, with a natural modification for $p = \infty$.

Definition :

- For $0 < p < \infty$, we will write $l^p(\widehat{G})$ for the space of all $H = H(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ such that

$$\|H\|_{l^p(\widehat{G})} = \left(\sum_{|\xi| \in \widehat{G}_0} d_\xi^{p(\frac{2}{p} - \frac{1}{2})} \|H(\xi)\|_{HS}^p \right)^{\frac{1}{p}} < \infty.$$

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- If $1 \leq p < \infty$ the quantity $\|H(\xi)\|_{l^p(\widehat{G})}$ defines a norm and $l^p(\widehat{G})$ endowed with it becomes a Banach space.
- If $0 < p < 1$ we can associate a Fréchet metric and the associated space becomes a complete metric space.

- Asymptotic properties : $n = dimG$

$$\sum_{\langle\xi\rangle\leq\lambda} d_\xi^2 \langle\xi\rangle^{rn} \asymp \lambda^{(r+1)n},$$

for $r > -1$. as $\lambda \to \infty$.

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- Convergence criterion :

$$\sum_{\langle\xi\rangle\leq\lambda} d_\xi^2 \langle\xi\rangle^{-s} < \infty \Leftrightarrow s > n$$

- Crucial reduction lemma
  All theorems on G/K can be reduced to the case of compact Lie groups. For $f \in C^\infty(G/K)$ its canonical lifting $\widetilde{f}$ is defined by $\widetilde{f}(yk) = f(y)$ for all $k \in K$ so that $\widetilde{f}$ is constant on the right cosets.

Lemma :

$$\widetilde{f} \in \mathbb{L}ip_G(\alpha, p) \Longleftrightarrow f \in \mathbb{L}ip_{G/k}(\alpha, p).$$

Lemma :

Let $H : \widehat{G} \to \bigcup_{d\in\mathbb{N}} \mathbb{C}^{d\times d}$. Be such that $H(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. For each ξ. Let $1 \leq \beta_0 < \infty$. Then

$$\langle\xi\rangle H(\xi) \in l^{\beta_0}(\widehat{G}) \Rightarrow H \in l^{\beta}(\widehat{G}).$$

for $\frac{n\beta_0}{\beta_0+n_0} < \beta < \infty$.

Theorem :

Let G be a compact Lie group of dimension n.
Let $0 < \alpha \leq 1$, $1 < p \leq 2$. and let $f \in \mathbb{L}ip_G(\alpha, p)$ then $\widehat{(I - \mathcal{L}_G)}^{1/2} f \in l^p(\widehat{G})$ for

$$\frac{n}{\alpha + n - \frac{n}{p} - 1} \leq \beta \leq \frac{p}{p-1}$$

consequently $\widehat{f} \in l^\gamma(\widehat{G})$ for $\frac{np}{\alpha p + n\gamma - n} \leq \gamma \leq \frac{p}{p-1}$.

Remark :

In the case $G = \mathbb{T}$

we have $f \in \mathbb{L}ip_G(\alpha, p) \Rightarrow (\widehat{(I - \Delta)^{1/2} f}) \in l^{\beta}(\mathbb{Z})$.

for $\frac{p}{p\alpha - 1} < \beta$.

$f \in \mathbb{L}ip_{\mathbb{T}^1}(\alpha, p) \Rightarrow \widehat{f} \in l^{\gamma}(\mathbb{Z})$.

for $\frac{p}{p\alpha + p - 1} < \gamma \leq \frac{p}{p-1}$.

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Duren's lemma :

suppose $c_k \geq 0$ and $0 < b < a$. Then $\sum_{k=1}^{N} k^a c_k = O(N^b)$ as $N \to \infty$.

$$\Leftrightarrow \sum_{k=N}^{\infty} c_k = O(N^{b-a})$$

as $N \to \infty$.

Theorem :

Let $0 < \alpha \leq 1$ and $f \in \mathbb{L}^2(G)$. then the conditions

$$f \in \mathbb{L}ip_G(\alpha, p),$$

and

$$\sum_{\tau \in \widehat{G}, \langle \xi \rangle \geq N} d_s \|\widehat{f}(\xi)\|_{HS}^2 = O(N^{-2\alpha})$$

as $N \to \infty$, are equivalents.

- Application to the regularity of Fourier multipliers on Hölder spaces :

Corollary :

Let $0 \leq \gamma < 1$ and let $a : \widehat{G} \to \bigcup_{d\in\mathbb{N}} \mathbb{C}^{d\times d}$ be such $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ for each ξ and $\|a(\xi)\|_{op} \leq C\langle\xi\rangle^{-\gamma}$.
Let A be the Fourier multiplier with symbol a, i.e,

$$\widehat{Af}(\xi) = a(\xi)\widehat{f}(\xi), \ \texttt{for all}\ \xi \in \widehat{G}.$$

Then $A : \mathbb{L}ip_G(\alpha, 2) \to \mathbb{L}ip_G(\alpha + \gamma, 2)$ is bounded for all α such that $0 < \alpha < 1 - \gamma$.

- Example : Lipschitz-Sobolev regularity for Bessel potential operators on compact lie groups.
  If $A = (I - \mathcal{L}_G)^{-\frac{\gamma}{2}}$, $0 \leq \gamma < 1$. we have

  $$\|(I - \mathcal{L}_G)^{-\frac{\gamma}{2}} f\|_{\mathbb{L}ip_G}(\alpha + \gamma, 2) \leq C\|f\|_{\mathbb{L}ip_G(\alpha,2)}$$

  for all α such that : $0 < \alpha < 1 - \gamma$.
  Hence :

  $$\|f\|_{\mathbb{L}ip_G(\alpha+\gamma,2)} \leq C\|(I - \mathcal{L}_G)^{-\frac{\gamma}{2}} f\|_{\mathbb{L}ip_G(\alpha,2)}.$$

- Sobolev-Lipschitz space
  For $0 \leq \gamma < 1$ and $0 < \alpha \leq 1 - \gamma$

$$H^{\gamma} Lip_G(\alpha, 2) = \{f \in D'(G) : (I - \mathcal{L}_G)^{\frac{\gamma}{2}} f \in Lip_G(\alpha, 2)$$

we have :

Corollary :

For every $0 \leq \gamma < 1$ and $0 < \alpha \leq 1 - \gamma$, we have the continuous embedding

$$H^{\gamma} Lip_G(\alpha, 2) \hookrightarrow Lip_G(\alpha + \gamma, 2).$$

- Younis-Dini-Lipschitz conditions

Theorem :

Let G/K be a compact homogenous manifold of dimension n.
Let $0 < \alpha \leq 1,\ d \in \mathbb{R},\ 1 < p \leq 2$ and suppose that

$$\|f(h.) - f(.)\|_{L^p(G/K)} = O(|h|^\alpha (log(\frac{1}{|h|})^d, \ \texttt{as}\ |h| \to 0.$$

Then we have $\widehat{(I - \mathcal{L}_{G/K})^{\frac{1}{2}}} \in l^\beta(\widehat{G}_0)$
provided that $\frac{n}{n+\alpha-\frac{n}{p}+1} < \beta \leq \frac{p}{p-1}$, $\texttt{for}\ d \leq 0.$
Consequently we have :

$$\widehat{f} \in l^\beta(\widehat{G}_0)\ \texttt{for}\ \frac{np}{\alpha p + np - n} < \gamma \leq \frac{p}{p-1}.$$

- Case $p = 2$

Theorem :

Let $\alpha \geq 0$ and $d \in \mathbb{R}$. Then

$$\|f(h.) - f(.)\|_{L^p(G/K)} = O(|h|^{\alpha}(log(\frac{1}{|h|})^{d}, \text{ as } |h| \to 0,$$

and

$$\sum_{[\xi]\in\widehat{G}_0, <\xi> \geq N} \|\widehat{f}(\xi)\|_{HS}^2 = O(N^{-2\alpha}(logN)^{2d}), \text{ as} N \to \infty,$$

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Thank you for your attention.