Reflection principles for functions of Neumann and Dirichlet Laplacians on open reflection invariant subsets

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(talk based on a joint work with Jacek Małecki)

Dedicated to Professor Thangavelu on the occasion of His 60th birthday
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setting

- Ω — a nonempty open subset of $\mathbb{R}^d$, $d \geq 1$, (a region)
- $\Delta = \sum_1^d \partial_j^2$ — the Laplacian
- $-\Delta_\Omega$ — the differential operator $f \mapsto -\Delta f$ with domain $C_c^\infty(\Omega)$
- $-\Delta_\Omega$ is symmetric,

$$\langle (-\Delta_\Omega) f, g \rangle_{L^2(\Omega)} = \langle f, (-\Delta_\Omega) g \rangle_{L^2(\Omega)}, \qquad f,\ g \in \mathrm{Dom}(-\Delta_\Omega),$$

and non-negative, $\langle (-\Delta_\Omega) f, f \rangle_{L^2(\Omega)} \geq 0$ for $f \in \mathrm{Dom}(-\Delta_\Omega)$.

Sobolev spaces

Sobolev space $H^n(\Omega)$ (denoted also $W^{n,2}(\Omega)$):

the linear space of functions $f \in L^2(\Omega)$ with the distributional derivatives $\partial^\alpha f$ in $L^2(\Omega)$ $\forall\ \alpha \in \mathbb{N}^d$, $|\alpha| \le n$, with the inner product

$$\langle f, g\rangle_{H^n(\Omega)} = \sum_{|\alpha|\le n} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2(\Omega)},$$

$H^n_0(\Omega)$ — the closure of $C_c^\infty(\Omega)$ in $(H^n(\Omega), \|\cdot\|_{H^n(\Omega)})$

$H^n(\Omega)$ and $H^n_0(\Omega)$ are Hilbert spaces

sesquilinear forms (general setting)

- $\mathfrak{t}$ — a sesquilinear form with (dense) domain $\mathrm{Dom}(\mathfrak{t})$ in $(\mathcal{H}, \langle\cdot,\cdot\rangle)$
- $A_{\mathfrak{t}}$ — the associated operator:

$$\mathrm{Dom}(A_{\mathfrak{t}}) = \{h \in \mathrm{Dom}(\mathfrak{t})\colon \exists u_h \in \mathcal{H}\ \forall h' \in \mathrm{Dom}(\mathfrak{t})\ \mathfrak{t}[h,h'] = \langle u_h, h'\rangle\}$$

$$A_{\mathfrak{t}} h = u_h, \qquad h \in \mathrm{Dom}(A_{\mathfrak{t}})$$

- if $\mathfrak{t}$ is Hermitian ($=$ symmetric) closed and non-negative, then $A_{\mathfrak{t}}$ is self-adjoint and non-negative.

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there is a canonical correspondence, on a given Hilbert space, between the family of lower semibounded closed and Hermitian sesquilinear forms and the family of lower semibounded self-adjoint operators.

definition of Neumann and Dirichlet Laplacians

$\mathfrak{t}_\Omega$ — the sesquilinear form defined on the domain $H^1(\Omega) \subset L^2(\Omega)$ by

$$\mathfrak{t}_\Omega[f,g] := \int_\Omega (\nabla f)(x) \cdot \overline{(\nabla g)(x)}\, dx = \sum_{j=1}^{n} \int_\Omega \partial_j f(x) \overline{\partial_j g(x)}\, dx.$$

The *Neumann Laplacian on* Ω, denoted $-\Delta_{N,\Omega}$, the operator on $L^2(\Omega)$ associated with the form $\mathfrak{t}_{N,\Omega} := \mathfrak{t}_\Omega$;
in particular, $\mathrm{Dom}(-\Delta_{N,\Omega}) \subset \mathrm{Dom}(\mathfrak{t}_{N,\Omega}) := H^1(\Omega)$

The *Dirichlet Laplacian on* Ω, denoted $-\Delta_{D,\Omega}$, the operator on $L^2(\Omega)$ associated with the form $\mathfrak{t}_{D,\Omega}$, the restriction of $\mathfrak{t}_\Omega$ to $H_0^1(\Omega)$;
in particular, $\mathrm{Dom}(-\Delta_{D,\Omega}) \subset \mathrm{Dom}(\mathfrak{t}_{D,\Omega}) := H_0^1(\Omega)$.

Neumann/Dirichlet Laplacians: comments

- $-\Delta_{D,\Omega}$ coincides with the Friedrichs extension of $\overline{-\Delta_\Omega}$
- $\mathrm{Dom}(-\Delta_{D,\Omega,}) = H^\Delta(\Omega) \cap H^1_0(\Omega)$, and

$$-\Delta_{D,\Omega} f = -\Delta f \ \text{ for } \ f \in \mathrm{Dom}(-\Delta_{D,\Omega});$$

  here $H^\Delta(\Omega) = \{f \in L^2(\Omega) \colon \Delta f \in L^2(\Omega)\}$ $(H^2(\Omega) \subset H^\Delta(\Omega))$
- much less is known about $\mathrm{Dom}(-\Delta_{N,\Omega})$ for general $\Omega \subset \mathbb{R}^d$
- if $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is open bounded with C^2 boundary, then there are much finer results concerning properties of $-\Delta_{D,\Omega}$ and $-\Delta_{N,\Omega}$; the D.L. refers to vanishing boundary values at $\partial\Omega$, the N.L. refers to vanishing directional derivatives at $\partial\Omega$
- the case $\Omega = \mathbb{R}^d$:
  $H^n(\mathbb{R}^d) = H^n_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \colon \|\cdot\|^n \mathcal{F} f \in L^2(\mathbb{R}^d)\}$,

$$\mathrm{Dom}(-\Delta_{D,\mathbb{R}^d}) = H^2(\mathbb{R}^d), \qquad -\Delta_{D,\mathbb{R}^d} f = -\Delta f$$

  and $H^1(\mathbb{R}^d) = H^1_0(\mathbb{R}^d)$ implies $-\Delta_{N,\mathbb{R}^d} = -\Delta_{D,\mathbb{R}^d}$.

- $0 \neq v \in \mathbb{R}^d$; σ_v — the orthogonal reflection in the hyperplane $\langle v\rangle^\perp$ perpendicular to v,

$$\sigma_v(x) = x - 2\frac{\langle v, x\rangle}{\|v\|^2}v, \qquad x \in \mathbb{R}^d.$$

(If $d = 1$, then the 'hyperplane' reduces to $\{0\}$, $\sigma_v(x) = -x$.)

- $\Omega \subset \mathbb{R}^d$ is symmetric in $\langle v\rangle^\perp$, if $\sigma_v(\Omega) = \Omega$
- the positive part of Ω: $\Omega_+ = \{\omega \in \Omega\colon \langle \omega, v\rangle > 0\}$
- $-\Delta_{N/D,\Omega}$ and $-\Delta_{N/D,\Omega_+}$ are non-negative and hence their spectra are contained in $[0, \infty)$.

Theorem (*J. Malecki–KS, 2017*)

Let $\Omega \subset \mathbb{R}^d$ — open symmetric in $\langle v\rangle^{\perp}$, Ω_+ — positive part of Ω, Φ — Borel function on $[0,\infty)$. Assume that $\Phi(-\Delta_{N,\Omega})$ is an integral operator with the kernel $K^{\Phi}_{-\Delta_{N,\Omega}}$. Then $\Phi(-\Delta_{N,\Omega_+})$ is also an integral operator with the kernel $K^{\Phi}_{-\Delta_{N,\Omega_+}}$ given by

$$K^{\Phi}_{-\Delta_{N,\Omega_+}}(x,y) = K^{\Phi}_{-\Delta_{N,\Omega}}(x,y) + K^{\Phi}_{-\Delta_{N,\Omega}}(\sigma_v(x),y), \qquad x,y \in \Omega_+.$$

Similarly, if $\Phi(-\Delta_{D,\Omega})$ is an i.o. with kernel $K^{\Phi}_{-\Delta_{D,\Omega}}$, then $\Phi(-\Delta_{D,\Omega_+})$ is also an i.o. with kernel $K^{\Phi}_{-\Delta_{D,\Omega_+}}$ given by

$$K^{\Phi}_{-\Delta_{D,\Omega_+}}(x,y) = K^{\Phi}_{-\Delta_{D,\Omega}}(x,y) - K^{\Phi}_{-\Delta_{D,\Omega}}(\sigma_v(x),y), \qquad x,y \in \Omega_+.$$

Corollary

Let $\Omega \subset \mathbb{R}^d$ — open symmetric in $\langle v \rangle^\perp$, Ω_+ — positive part of Ω and let p_t^{N,Ω_+} and p_t^{D,Ω_+}, and $p_t^{N,\Omega}$ and $p_t^{D,\Omega}$, denote the Neumann and the Dirichlet heat kernels on Ω_+ and Ω, respectively. Then

$$p_t^{N,\Omega_+}(x,y) = p_t^{N,\Omega}(x,y) + p_t^{N,\Omega}(\sigma_v(x),y), \qquad x,y \in \Omega_+, \quad t > 0.$$

and

$$p_t^{D,\Omega_+}(x,y) = p_t^{D,\Omega}(x,y) - p_t^{D,\Omega}(\sigma_v(x),y), \qquad x,y \in \Omega_+, \quad t > 0.$$

even and odd parts of $F\colon \Omega \to \mathbb{C}$ (w.r.t. the v direction):

$$F_{\mathrm{even/odd}}(x) = (F(x) \pm F(\sigma_v x))/2;$$

we theat them as functions on Ω_+.

Lemma

$$H^1(\Omega_+) = \big(H^1(\Omega)\big)_{\mathrm{even}} \qquad \text{and} \qquad H^1_0(\Omega_+) = \big(H^1_0(\Omega)\big)_{\mathrm{odd}}.$$

Proposition

$$\mathrm{Dom}(-\Delta_{N,\Omega_+}) = \big(\mathrm{Dom}(-\Delta_{N,\Omega})\big)_{\mathrm{even}}$$

and

$$(-\Delta_{N,\Omega_+})(F_{\mathrm{even}}) = \big((-\Delta_{N,\Omega})F\big)_{\mathrm{even}}, \quad \text{for} \quad F \in \mathrm{Dom}(-\Delta_{N,\Omega}).$$

Similarly,

$$\mathrm{Dom}(-\Delta_{D,\Omega_+}) = \big(\mathrm{Dom}(-\Delta_{D,\Omega})\big)_{\mathrm{odd}}$$

and

$$(-\Delta_{D,\Omega_+})(F_{\mathrm{odd}}) = \big((-\Delta_{D,\Omega})F\big)_{\mathrm{odd}}, \quad \text{for} \quad F \in \mathrm{Dom}(-\Delta_{D,\Omega}).$$

outline of proof of main theorem

Ψ — an arbitrary Borel function on $\mathbb{R}$

- **commuting property of the spectral functional calculus**: if A is self-adjoint operator on $\mathcal{H}$ and $B \in \mathcal{B}(\mathcal{H})$ is such that $BA \subset AB$, then also $B\Psi(A) \subset \Psi(A)B$
- **two-Hilbert space and two-operator version of the above**: if A_1 and A_2 are self-adjoint operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $B\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator such that $BA_1 \subset A_2B$, then also $B\Psi(A_1) \subset \Psi(A_2)B$.

If Ψ is bounded, then $\Psi(A), \Psi(A_i)$ are bounded and the concluding inclusions become identities.

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If Ψ is bounded, then $\Psi(A), \Psi(A_i)$ are bounded and the concluding inclusions become identities.

[*a folklore, but it is hard to find it in the literature in the above formulation; there is, however, Berberian's trick that allows to change the intertwining condition into the commuting condition: take $A_1 \oplus A_2$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $(x, y) \mapsto (0, Bx)$ as the bounded operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$*]

We write $-\Delta$, $-\Delta_+$, K^Φ instead of $-\Delta_{N,\Omega}$, $-\Delta_{N,\Omega_+}$ and $K^\Phi_{-\Delta_{N,\Omega}}$.

The mapping $L^2(\Omega) \ni F(x) \to \check{F}(x) := F(\sigma_v(x)) \in L^2(\Omega)$, leaves $\mathrm{Dom}(-\Delta)$ invariant, and $(-\Delta F)\check{} = -\Delta\check{F}$, $F \in \mathrm{Dom}(-\Delta)$. Hence, for $F \in \mathrm{Dom}(\Phi(-\Delta))$ and $x, y \in \Omega$,

$$(\Phi(-\Delta)F)\check{} = \Phi(-\Delta)\check{F} \Rightarrow K^\Phi(\sigma_v(x), y) = K^\Phi(x, \sigma_v(y))$$

By Proposition

$$(-\Delta_+)(F_{\mathrm{even}}) = (-\Delta F)_{\mathrm{even}}, \qquad F \in \mathrm{Dom}(-\Delta)$$

and hence, (we take $\mathcal{H}_1 = L^2(\Omega)$, $\mathcal{H}_2 = L^2(\Omega_+)$, $A_1 = -\Delta$, $A_2 = -\Delta_+$, and $B\colon L^2(\Omega) \to L^2(\Omega_+)$ defined by $F \mapsto F_{\mathrm{even}}$,

$$\Phi(-\Delta_+)(F_{\mathrm{even}}) = (\Phi(-\Delta)F)_{\mathrm{even}}, \qquad F \in \mathrm{Dom}(\Phi(-\Delta)).$$

[recall: $\Phi(-\Delta_+)(F_{\rm even}) = (\Phi(-\Delta)F)_{\rm even}$]
Given $f \in \mathrm{Dom}(\Phi(-\Delta_+))$ with $F \in \mathrm{Dom}(\Phi(-\Delta))$ such that $F_{\rm even} = f$, we can assume that F is even and then for $x \in \Omega_+$ we obtain

$$\begin{aligned}\Phi(-\Delta_+)f(x) &= \frac{1}{2}\big(\Phi(-\Delta)F(x) + \Phi(-\Delta)F(\sigma_v(x))\big)\\ &= \frac{1}{2}\Big(\int_\Omega K^\Phi(x,y)F(y)\,dy + \int_\Omega K^\Phi(\sigma_v(x),y)F(y)\,dy\Big)\\ &= \frac{1}{2}\Big(\int_{\Omega_+}\Big[K^\Phi(x,y) + K^\Phi(x,\sigma_v(y))\Big]f(y)\,dy +\\ &+ \int_{\Omega_+}\Big[K^\Phi(\sigma_v(x),y) + K^\Phi(\sigma_v(x),\sigma_v(y))\Big]f(y)dy\Big)\\ &= \int_{\Omega_+}\big[K^\Phi(x,y)f(y) + K^\Phi(x,\sigma_v(y))\big]f(y)\,dy,\end{aligned}$$

remark:
$K^\Phi(\sigma_v(x),y) = K^\Phi(x,\sigma_v(y)) \Rightarrow K^\Phi(\sigma_v(x),\sigma_v(y)) = K^\Phi(x,y)$

resolvents, Riesz potentials, Green's functions

Applications of the main Theorem, apart of heat kernels, include

- resolvents, $\Phi_\lambda(u) = (u + \lambda)^{-1}$, $\lambda > 0$, $\mathcal{R}_\lambda = (-\Delta_{N/D,\Omega} + \lambda I)^{-1}$
- Riesz potentials, $\Phi_\sigma(u) = u^{-\sigma}$, $\sigma > 0$, $R_\sigma = (-\Delta_{N/D,\Omega})^{-\sigma}$
- $\sigma = 1$ (Newtonian potential operator) or the limiting case $\lambda = 0$ for the resolvent operators $\mathcal{R}_\lambda$ (Green's function)

associated to the Neumann/Dirichlet Laplacians on regions of $\mathbb{R}^d$. These conclude in reflection principle formulas for the corresponding integral kernels, i.e. resolvent kernels, Riesz potential kernels, Green's functions.

It is interesting to recover *reflection formulas* for heat kernels for several regions by using formulas for Neumann and Dirichlet heat kernels for these regions given in terms of series.
Let $\Phi \in (0, 2\pi]$ and let Ω_Φ denote the open (infinite) cone

$$\Omega_\Phi = \{x = \rho e^{i\theta} \in \mathbb{R}^2 \colon 0 < \rho < \infty, \quad 0 < \theta < \Phi\}$$

on the plane with vertex at the origin and aperture Φ. By $p_t^{N/D,\Phi}$ we denote the Neumann/Dirichlet heat kernels related to Ω_Φ.

An old Carslaw and Jaeger formula that expresses $p_t^{D,\Phi}(x,y)$ by the convergent series,

$$p_t^{D,\Phi}(x,y) = \frac{1}{2\Phi t}\exp\left(-\frac{\rho^2+r^2}{4t}\right)\sum_{j=1}^{\infty} I_{\pi j/\Phi}\left(\frac{\rho r}{2t}\right) 2\sin\left(j\frac{\pi}{\Phi}\theta\right)\sin\left(j\frac{\pi}{\Phi}\eta\right),$$

where $x=\rho e^{i\theta}\in\Omega_\Phi$, $y=re^{i\eta}\in\Omega_\Phi$ (I_ν is the modified Bessel function of order ν), leads, after a calculation and with use of

$$B^\Phi(\tau,\gamma)+B^\Phi(\tau,\Phi-\gamma)=2B^{\Phi/2}(\tau,\gamma),$$

where $B^\Phi(\tau,\gamma)=\sum_{j=1}^\infty I_{\pi j/\Phi}(\tau)\cos\left(j\frac{\pi}{\Phi}\gamma\right)$, to the formula

$$p_t^{D,\Phi/2}(x,y)=p_t^{D,\Phi}(x,y)-p_t^{D,\Phi}(\tilde{x},y), \qquad x,y\in\Omega_{\Phi/2},$$

where $\tilde{x}=\rho e^{i(\Phi-\theta)}$ denotes the reflection of $x=\rho e^{i\theta}$ with respect to the bisector of the cone Ω_Φ.

Dziękuję!

Thanks! $\varepsilon\upsilon\chi\alpha\rho\iota\sigma\tau\omega$! Danke! Gracias! Ačiu! Merci! Gracie! Děkuji! Kiitos! Спасибо! Eskerrik asko! Hvala! Tack! ধন্যবাদ धन्यवाद شکریه நன்றி ಧನ್ಯವಾದ


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something special

Nothing can ever happen twice.
In consequence, the sorry fact is
that we arrive here improvised
and leave without the chance to practice.

Even if there is no one dumber,
if you're the planet's biggest dunce,
you can't repeat the class in summer:
this course is only offered once.

No day copies yesterday,
no two nights will teach what bliss is
in precisely the same way,
with precisely the same kisses.

One day, perhaps some idle tongue
mentions your name by accident:
I feel as if a rose were flung
into the room, all hue and scent.

something special is continued

The next day, though you're here with me,
I can't help looking at the clock:
A rose? A rose? What could that be?
Is it a flower or a rock?

Why do we treat the fleeting day
with so much needless fear and sorrow?
It's in its nature not to stay:
Today is always gone tomorrow.

With smiles and kisses, we prefer
to seek accord beneath our star,
although we're different (we concur)
just as two drops of water are.

Wisława Szymborska, 1923 - 2012
Nobel Prize in Literature, 1996