Some recent progresses on noncommutative ergodic theory

Guixiang HONG, Wuhan University
Based on joint work with Benben Liao and Simeng Wang

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Birkhoff ergodic theorem

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $T : \Omega \to \Omega$ be a measurable measure preserving transformation, which is ergodic, that is, $T^{-1}E = E$ implies $\mu(E) = 0$ or 1
- Birkhoff's pointwise ergodic theorem : $\forall f \in L^1(\Omega)$,

$$M_n f(x) := \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) \to \int_\Omega f d\mu, \ a.e. x \in \Omega$$

- When T is not ergodic, then we have

$$M_n f(x) := \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) \to E(f|inv)(x), \ a.e. x \in \Omega$$

where *inv* is the σ-algebra generated by

$$\{B \in \mathcal{F} : T^{-1}B = B\}.$$

(Now $\mu(\Omega) = 1$ does not play any role.)

Three easy Remarks

- Two-sided Birkhoff :

$$M_n f(x) = \frac{1}{2n+1} \sum_{k=-n}^{n} f(T^k x) \to E(f|inv)(x), \ a.e.x \in \Omega$$

- Continuous Birkhoff :

$$M_t f(x) = \frac{1}{2t} \int_{-t}^{t} f(T^s x) ds \to E(f|inv)(x), \ a.e.x \in \Omega$$

- Birkhoff's maximal ergodic theorem :

$$\| \sup_{t>0} |M_t f| \|_{1,\infty} \lesssim \|f\|_1 .$$

Two inspiring remarks

- When $\Omega = \mathbb{R}$, T^s is the translation, M_t becomes the averages over the ball $B(0,t)$
- The first observation. Calderón's transference principle :

$$\|\sup_{t>0}|A_t f|\|_{L^{1,\infty}(\mathbb{R})} \leq C\|f\|_{L^1(\mathbb{R})} \Rightarrow \|\sup_{t>0}|M_t f|\|_{L^{1,\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}$$

- The second observation. Wiener's ergodic theorem : considering d commuting measurable flows

$$\|\sup_{t>0}|M_t f|\|_{L^{1,\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}$$

where

$$M_t f(x) = \frac{1}{|B(0,t)|}\int_{B(0,t)} f(T_1^{s_1}\cdots T_d^{s_d}x)ds.$$

Framework I

- A measurable dynamical system is a quadruple (Ω, μ, G, T), where G is a locally compact second countable (lcsc) group, T is a continuous action of G on (Ω, μ).
- Denote by $P(G)$ the set of probability measures on G. For each $v \in P(G)$, there corresponds an operator $T(v)$, with norm bounded by 1 in every $L^p(\Omega)$, $1 \leq p \leq \infty$, given by

$$T(v)f(x) = \int_G f(T_g x)dv(g), \ \forall f \in L^p(\Omega).$$

Framework II

Let $t \to \nu_t$ be a weakly continuous map from $\mathbb{R}_+$ to $P(G)$, namely $t \to \nu_t(f)$ is continuous for each $f \in C_c(G)$. We will refer to $(\nu_t)_{t>0}$ as a one-parameter family of probability measures. We can now formulate the following.

Definition

A one-parameter family $(\nu_t)_{t>0} \subset P(G)$ will be called a pointwise ergodic family in L^p if for every measure-preserving continuous action T of G on any (Ω, μ), and for any $f \in L^p(\Omega)$, we have $T(\nu_t)f$ converges a.e. as t tend vers infinity.

Framework III

- Let G lcsc, d be a admmisible metric, B_t (resp. S_t) be the corresponding balls (resp. spheres) of radius t and center e and let β_t (resp. σ_t) be the normalized ball (resp. sphere) average.
- Ball averaging problem : When is $(\beta_t)_{t>0}$ a pointwise ergodic family in L^p $(1 \leq p \leq \infty)$?
- Sphere averaging problem : When is $(\sigma_t)_{t>0}$ a pointwise ergodic family in L^p $(1 \leq p \leq \infty)$?

Ball averaging problems

Polynomial growth groups I

- Doubling condition : $m_G(B_{2t}) \leq C(G,d)m_G(B_t)$, $\forall t > 0$
- Asymptotical invariance : $\forall g \in G$,

$$\lim_{t\to\infty} \frac{m_G((B_t g)\Delta B_t)}{m_G(B_t)} = 0$$

- Polynomial growth : $G = \bigcup_{n\geq 0} V^n$ where V is a symmetric cpt set and $V^n = V \cdot V \cdots V$; $|g|_V = \min\{n : g \in V^n\}$. G is called of polynomial volumn growth if

$$\lim_{n\to\infty} \frac{\log m_G(V^n)}{n} =: h_V = 0 \text{ and } \limsup_{n\to\infty} \frac{\log m_G(V^n)}{\log n} =: q_V < \infty.$$

Polynomial growth groups II

Pointwise ergodic theorem for groups with volumn doubling and asymptotically invariant balls.

Theorem (Wiener, Riesz, Calderón etc)

Let G be lcsc group with invariant metric d. If the corresponding balls B_t satisfies volumn doubling condition and is asymptotically invariant, then $(\beta_t)_t$ is a pointwise ergodic family in L^p for all $1 \leq p \leq \infty$.

Pointwise ergodic theorem for groups with polynomial volumn growth with word metrics.

Theorem (Pansu, Breuillard, Guivarch, Tessera, Losert, Gromov etc)

Let G be lcsc of polynomial growth w.r.t a word metric d. Then $(\beta_n)_n$ is a pointwise ergodic family in L^p for all $1 \leq p \leq \infty$.

Sphere averaging problems

Euclidean spherical averages

Theorem (Stein, Bourgain, Calderón)

Let $(\sigma_t)_t$ be the normalized measure over the sphere on $\mathbb{R}^n$ $(n \geq 2)$ of radius t and center 0. Then for any group action T on any measure space (Ω, μ), for any $f \in L^p(\Omega)$ with $p > \frac{n}{n-1}$

$$\|\sup_{t>0} T(\sigma_t)f\|_p \leq C_p\|f\|_p.$$

Theorem (Jones)

$(\sigma_t)_t$ is a pointwise ergodic family in L^p for all $p > \frac{n}{n-1}$.

$\mathbb{C}^n$-spheres on the Heisenberg groups

Theorem (Nevo, Thangavelu, Narayanan)

The normalized measures $(\sigma_t)_t$ over the spheres $S_t \subset \mathbb{C}^n \subset H^n := \mathbb{C}^n \times \mathbb{R}$ $(n \geq 2)$ of radius t and center 0 is a pointwise ergodic family in L^p for $p > \frac{2n}{2n-1}$.

- Nevo-Thangavelu : spectral method to deal with maximal inequality ($p > \frac{2n-1}{2n-2}$) and spetral method to prove pointwise ergodic theorem
- Narayanan-Thangavelu : establish the maximal inequality for $p > \frac{2n}{2n-1}$, then use Calderón's transference principle

Noncommutative ergodic theorems

Noncommutative L_p-spaces

- Let $\mathcal{M}$ be a semifinite Von Neumann algebra with a normal faithful trace τ and $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^{+}$ which is the set of all positive x in $\mathcal{M}$ such that $\tau(suppx) < \infty$.
- Let $0 < p < \infty$. We define

$$\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}, \qquad \forall x \in S_{\mathcal{M}}$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the modulus of x. We denote the completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ as $L_p(\mathcal{M})$ which is called the non-commutative L_p space associated with $(\mathcal{M}, \tau)$. For convenience, we usually set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$.

The space $L_p(\mathcal{M};\ell_\infty)$

- The norm of $x=(x_n)_n$ in $L_p(\mathcal{M};\ell_\infty)$ is given by

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)}=\inf\Big\{\|a\|_{2p}\sup_{n\geq 1}\|y_n\|_\infty\|b\|_{2p}\Big\},$$

  where the infimum runs over all factorizations of x of the following form : there exist $a,b\in L_{2p}(\mathcal{M})$ and a bounded sequence $y=(y_n)$ in $L_\infty(\mathcal{M})$ such that

$$x_n=ay_nb,\quad\forall\ n.$$

- $$\|x\|_{L_p(\mathcal{M};\ell_\infty)}=\Big\|{\sup_n}^+ x_n\Big\|_p$$

Almost uniformly convergence

- Let $(x_\lambda)_{\lambda\in\Lambda}$ be a family of elements in $L_p(\mathcal{M})$. Recall that $(x_\lambda)_{\lambda\in\Lambda}$ is said to converge almost uniformly to x, abbreviated by $x_\lambda \xrightarrow{a.u} x$, if for every $\epsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that

$$\tau(1-e) < \epsilon \quad \text{and} \quad \lim_\lambda \|e(x_\lambda - x)\|_\infty = 0.$$

- Also, $(x_\lambda)_{\lambda\in\Lambda}$ is said to converge bilaterally almost uniformly to x, abbreviated by $x_\lambda \xrightarrow{b.a.u} x$, if for every $\epsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(1-e) < \epsilon \quad \text{and} \quad \lim_\lambda \|e(x_\lambda - x)e\|_\infty = 0.$$

Framework I

- Noncommutative case : $(\mathcal{M}, \tau, G, \alpha)$ is called a W^*-dynamical system if $\alpha : G \to Aut(\mathcal{M})$ is a continuous group homomorphism in the pointwise weak* topology and α is trace preserving, that is

$$\tau(\alpha(g)x) = \tau(x)$$

  for all $g \in G$ and $x \in L_1(\mathcal{M}) \cap \mathcal{M}$.
- For each $\mu \in P(G)$, the set of probability measures on G, there corresponds an operator $\alpha(\mu)$, with norm bounded by 1 in every $L_p(\mathcal{M})$, $1 \leq p \leq \infty$, given by

$$\alpha(\mu)x = \int_G \alpha(g)x d\mu(g), \; \forall x \in L_p(\mathcal{M}).$$

Framework II

Let $t \to v_t$ be a weakly continuous map from $\mathbb{R}_+$ to $P(G)$, namely $t \to v_t(f)$ is continuous for each $f \in C_c(G)$. We will refer to $(v_t)_{t \geq 0}$ as a one-parameter family of probability measure. We can now formulate the following.

Definition

A one-parameter family $(v_t)_{t>0} \subset P(G)$ will be called a noncommutative pointwise ergodic family in L_p if for every W^*-dynamical system $(\mathcal{M}, \tau, G, \alpha)$ and every $x \in L_p(\mathcal{M})$, $\alpha(v_t)x$ converge almost uniformly to $F(x)$ as t tend to ∞.

Nc Birkhoff

Theorem (Yeadon, JLMS 77')

The one-parameter family $(\mu_t)_{t>0} \subset P(\mathbb{R})$ *is a noncommutative pointwise ergodic family in* L_1*, where*

$$\mu_t = \frac{1}{2t}\int_{-t}^{t} \delta_s ds = \frac{1}{t}\int_0^t \frac{1}{2}(\delta_s + \delta_{-s})ds.$$

Theorem (Junge-Xu, JAMS 07')

The one-parameter family $(\mu_t)_{t>0} \subset P(\mathbb{R})$ *is a noncommutative pointwise ergodic family in* L_p *for* $1 < p < \infty$.

Actually, Yeadon and Junge/Xu proved the Noncommutative Dunford-Schwartz (as well as Stein's) ergodic theorem for semigroups, which implies immediately noncommutative Birkhoff ergodic theorem.

Nc ergodic theorems for groups of polynomial volumn growth I

Theorem (H-Liao-Wang)

Let (G, d) satisfy the doubling condition. Then $(\beta_t)_t$ is a noncommutative pointwise ergodic family in L^p for all $1 \leq p \leq \infty$.

Theorem (H-Liao-Wang)

Let G be lcsc of polynomial growth w.r.t a word metric d. Then $(\beta_n)_n$ is a noncommutative pointwise ergodic family in L^p for all $1 \leq p \leq \infty$.

Comments on the proof

- Random dyadic cubes (to get operator-valued maximal inequality)+ Noncommutative transference (to get maximal ergodic theorems) + nc Banach principle (to get individual ergodic theorems)
- The strong (p, p) type transference principle holds not only for automorphisms but for all actions satisfying the following three conditions ; the weak (p, p) type transference principle does need the action to be automorphisms. But in the case, we provide another method based on Markov random walk to get the maximal ergodic inequalities.

Nc ergodic theorems for groups of polynomial volumn growth cts

Let $(\mathcal{M},\tau)$ be as before. For a fixed $1 \leq p \leq \infty$, we will be interested in actions $\alpha = (\alpha_g)_{g\in G}$ on $L_p(\mathcal{M})$ with the following conditions :

($\mathbf{A}_1^p$) Continuity : for all $x \in L_p(\mathcal{M})$, the map $g \mapsto \alpha_g x$ from G to $L_p(\mathcal{M})$ is continuous. Here we take the norm topology on $L_p(\mathcal{M})$ if $1 \leq p < \infty$ and the w*-topology if $p = \infty$.

($\mathbf{A}_2^p$) Uniform boundedness : $\sup_{g\in G} \|\alpha_g : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| < \infty$.

($\mathbf{A}_3^p$) Positivity : for all $g \in G$, $\alpha_g x \geq 0$ if $x \geq 0$ in $L_p(\mathcal{M})$.

Nc ergodic theorems for groups of polynomial volumn growth II

Theorem (H-Liao-Wang)

Fix $1 < p < \infty$. Let $\alpha = (\alpha_g)_{g \in G}$ be an action on $L_p(\mathcal{M})$ which satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$.

- Let (G, d) satisfy the doubling condition. Then there exists a dyadic subsequence $(r_k)_{k \geq 1}$ with $2^k \leq r_k < 2^{k+1}$ such that $(A_{r_k} x)_{k \geq 1}$ converges b.a.u. to Px for all $x \in L_p(\mathcal{M})$. If additionally $p \geq 2$, $(A_{r_k} x)_{k \geq 1}$ converges a.u. to Px for all $x \in L_p(\mathcal{M})$.
- Let G be lcsc of polynomial growth w.r.t a word metric d. Then $(A_n x)_{n \geq 1}$ convergences b.a.u. to Px for all $x \in L_p(\mathcal{M})$. Moreover if G is discrete, finitely generated and nilpotent and if $p \geq 2$, then $(A_n x)_{n \geq 1}$ converges a.u to Px.

A corollary

Corollary (H-Liao-Wang)

Fix $1 < p < \infty$. Let $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive invertible operator with positive such that $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. Denote

$$A_n = \frac{1}{2n+1} \sum_{k=-n}^{n} T^k, \quad n \in \mathbb{N}.$$

Then $(A_n x)_{n\geq 1}$ converges b.a.u to Px for all $x \in L_p(\mathcal{M})$. If additionally $p \geq 2$, $(A_n x)_{n\geq 1}$ converges a.u to Px.

Remark

Note that the above result is not true for $p = 1$, even for positive invertible isometries on classical L_1-spaces. So it is natural to assume $p \neq 1$ in the above results.

Nc Jones and Nc Nevo-Thangavelu

Theorem (H)

$(\sigma_t)_t$ is a nc pointwise ergodic family in L^p for all $p > \frac{n}{n-1}$.

Theorem (H)

The normalized measures $(\sigma_t)_t$ over the spheres $S_t \subset \mathbb{C}^n \subset H^n := \mathbb{C}^n \times \mathbb{R}$ $(n \geq 2)$ of radius t and center 0 is a pointwise ergodic family in L^p for $p > \frac{2n-1}{2n-2}$.

Theorem (H)

The normalized measures $(\bar{\sigma}_t)_t$ over the spheres $\bar{S}_t \subset \mathbb{C}^n \subset \bar{H}^n$ $(n \geq 2)$ of radius t and center 0 is a noncommutative pointwise ergodic family in L^p for $p > \frac{2n}{2n-1}$.

Thanks for your attention