Sidon sets in discrete groups

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Sidon sets in the last century (Golden age)

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n \in \Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n \in \Lambda} |a_n| < \infty$$

Sidon sets (and more generally **"thin** sets" e.g. **Helson** sets) were a very active subject in the 1960's and 1970's: Kahane, Malliavin, Varopoulos, Yves Meyer, Bonami +others (in France), Edwards & Gaudry (Australia), Figa-Talamanca (Italy), Rudin, Hewitt & Ross, Rider (USA), Hartman & Ryll-Nardzewski, Bożejko (Poland), Katznelson (Israel), Herz, Drury (Canada)...
The first period culminated with Sam Drury's solution of

"the Union problem":

Drury (1970): The union of two Sidon sets is again a Sidon set.

References

1970: E. Hewitt and K. Ross, *Abstract harmonic analysis, Volume II, Structure and Analysis for Compact Groups, Analysis on Locally Compact Abelian Groups*, Springer, Heidelberg, 1970.

1975: J. López and K.A. Ross, *Sidon sets.* Lecture Notes in Pure and Applied Mathematics, Vol. 13. Marcel Dekker, Inc., New York, 1975.

1981: M.B. Marcus and G. Pisier, *Random Fourier series with Applications to Harmonic Analysis*, Annals of Math. Studies n°101, Princeton Univ. Press, 1981.

1985: J. P. Kahane, *Some random series of functions. Second edition*, Cambridge University Press, 1985.

2013: C. Graham and K. Hare, *Interpolation and Sidon sets for compact groups*. Springer, New York, 2013. xviii+249 pp.

More recent work 2015-2017: J. Bourgain-M.Lewko Annales Inst. Fourier 2017

G.P. Math. Res. Letters 2017 + papers to appear, all on arxiv

Sidon

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n\in\Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

Equivalently: $\exists C$ such that $\forall A \subset \Lambda$ with $|A| < \infty$

$$\sum_{n\in A} |a_n| \leq C \| \sum_{n\in A} a_n e^{int} \|_\infty$$

More generally, let G be a compact Abelian group, $\Lambda = \{\varphi_n\} \subset \widehat{G}$ (characters on G), Λ is Sidon if $\exists C$ such that $\forall A$ with $|A| < \infty$

$$\sum_{n\in A} |a_n| \leq C \| \sum_{n\in A} a_n \varphi_n \|_\infty$$

Fundamental Example

$$G = \mathbf{T}^{\mathbb{N}}$$

$$\forall z = (z_n) \in \mathbf{T}^{\mathbb{N}} \quad \varphi_n(z) = z_n$$

$$\| \sum a_n \varphi_n \|_\infty = \sum |a_n| \quad (C = 1)$$

Note: (φ_n) are independent random variables

More Examples

Hadamard lacunary sequences $n_1 < n_2 < \cdots < n_k, \cdots$ such that

$$\inf_k \frac{n_{k+1}}{n_k} > 1$$

Explicit example

$$n_k = 2^k$$

Basic Example: Quasi-independent sets

Λ is quasi-independent if all the sums

$$\{\sum_{n\in A} n \mid A \subset \Lambda, |A| < \infty\} \text{ are distinct numbers}$$

$$\text{quasi-independent} \Rightarrow \text{Sidon}$$

Main Open Problem

Is every **Sidon** set a finite union of **quasi-independent** sets ?

The recent rebirth

Bourgain and Lewko (Ann. Inst. Fourier 2017) wondered whether a group environment is needed for the known results about Sidon sets

Question

What remains valid if $\Lambda \subset \widehat{G}$ is replaced by a *uniformly bounded* orthonormal system ?

Let $\Lambda = \{\varphi_n\} \subset L_\infty(T, m)$ orthonormal in $L_2(T, m)$ ((T, m) any probability space)

(i) We say that (φ_n) is Sidon with constant C if for any N and any complex sequence (a_n) we have

$$\sum_1^N |a_n| \le C \| \sum_1^N a_n \varphi_n \|_\infty .$$

(ii) Let $k \ge 1$. We say that (φ_n) is $\otimes^k$-Sidon with constant C if the system $\{\varphi_n(t_1) \cdots \varphi_n(t_k)\}$ (or equivalently $\{\varphi_n^{\otimes k}\}$) is Sidon with constant C in $L_\infty(T^k, m^{\otimes k})$.

Crucial remark: For characters on a compact group T

$$\text{Sidon} \Leftrightarrow \otimes^k - \text{Sidon}$$

because

$$\| \sum_1^N a_n \varphi_n \|_\infty = \| \sum_1^N a_n \varphi_n(t_1) \cdots \varphi_k(t_k) \|_{L_\infty(T^k)}$$

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Theorem (Union problem for unif.bded o.n. systems)

Let (φ_n) be an orthonormal system bounded in L_∞. Assume that (φ_n) is the union of two (or finitely many) Sidon systems. Then (φ_n) is $\otimes^4$-Sidon.

But is it Sidon ?

The Return of Union Problem

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But is it Sidon ?

No !

Let (ε_n) be i.i.d. ± 1-valued symmetric random variables (e.g. the Rademacher functions)

Proposition

There are two orthonormal martingale difference sequences (φ_n^+) and (φ_n^-) in L_2 with $\mathrm{span}[\varphi_n^+] \perp \mathrm{span}[\varphi_n^-]$ such that

$$(\varphi_n^+) = (\varphi_n^+) = (\varepsilon_n) \text{ in distribution}$$

but their union is not a Sidon system.
More precisely the union of $\{\varphi_k^+ \mid k \le n\}$ and $\{\varphi_k^- \mid k \le n\}$ has a Sidon constant $C_n \approx \sqrt{n}$.

Same holds with (ε_n) replaced by our fundamental example (z_n)

About Randomly Sidon

We say that (φ_n) is randomly Sidon with constant C if for any N and any complex sequence (a_n) we have

$$\sum_1^N |a_n| \leq C \text{Average}_{\pm 1} \| \sum_1^N \pm a_n \varphi_n \|_\infty,$$

Theorem

Let (φ_n) be an orthonormal system system bounded in L_∞. The following are equivalent:

(i) *The system (φ_n) is randomly Sidon.*

(ii) *The system (φ_n) is $\otimes^4$-Sidon.*

(iii) *The system (φ_n) is $\otimes^k$-Sidon for all $k \geq 4$.*

(iv) *The system (φ_n) is $\otimes^k$-Sidon for some $k \geq 4$.*

This generalizes Rider's 1975 result that randomly Sidon implies Sidon for characters

Open question: What about $k = 2$ or $k = 3$?

Non-commutative case

Two different possibilities have been considered
EITHER
(I) Replace the compact Abelian group (e.g. $\mathbf{T}$) by a *non-Abelian compact group* G such as $SO(n), SU(n), U(n), \ldots$.
Then the set $\Lambda \subset \widehat{G}$ is a subset of the dual object i.e. the set of unitary irreducible representations
OR
(II) Replace the discrete Abelian group (e.g. $\mathbb{Z}$) by a *non-Abelian discrete group* Γ such as a free group $\mathbb{F}_n$
Then $\Lambda \subset \Gamma$
In both cases I have obtained the analogues of the preceding i.e. results for general orthonormal functions, that imply the case of characters as special case using the notion of $\otimes^k$-Sidon

Sidon sets in duals of compact non-commutative groups

G compact non-commutative group
$\widehat{G}$ the set of distinct irreps, $d_\pi = \dim(H_\pi)$
$\Lambda \subset \widehat{G}$ is called Sidon if $\exists C$ such that $\forall a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C \|\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}(\pi a_\pi)\|_\infty .$$

$\Lambda \subset \widehat{G}$ is called randomly Sidon if $\exists C$ such that $\forall a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C\mathbb{E} \|\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}(\varepsilon_\pi \pi a_\pi)\|_\infty$$

where (ε_π) are an independent family such that each ε_π is uniformly distributed over $O(d_\pi)$.
Important Remark (easy proof) Different randomizations (e.g. Gaussian random matrices) lead to equivalent definitions

Fundamental example

$$G = \prod_{n \geq 1} U(d_n)$$

$$\Lambda = \{\pi_n \mid n \geq 1\}$$

$$\pi_n : \;\; G \to U(d_n) \quad n\text{-th coordinate}$$

$$C = 1 : \;\; \sum\nolimits_{n \geq 1} d_n \mathrm{tr}|a_n| = \| \sum\nolimits_{n \geq 1} d_n \mathrm{tr}(\pi_n a_n) \|_\infty .$$

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Observe that for the functions $\varphi_n(i,j)$ defined on (G, m_G) by

$$\varphi_n(i,j)(g) = \pi_n(g)_{ij}$$

$\{d_n^{1/2} \varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$ is an orthonormal system.

Rider (1975, unpublished) extended all results previously mentioned to arbitrary compact groups
in particular: randomly Sidon implies Sidon (solving the non-commutative union problem)
I posted a paper on this on arxiv including (presumably) his proof

General matricial systems

Assume given a sequence of finite dimensions d_n.
For each n let (φ_n) be a random matrix of size $d_n \times d_n$ on (T, m).
We call this a "matricial system":

$$\varphi_n = [\varphi_n(i,j)]$$

or rather for $t \in T$

$$\varphi_n(t) = [\varphi_n(i,j)(t)]$$

The **uniform boundedness condition** becomes

$$\exists C' \ \forall n \quad \|\varphi_n\|_{L_\infty(M_{d_n})} \leq C'.$$

The **orthonormality condition** becomes :

$$\{d_n^{1/2}\varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$$

is an orthonormal system.

The definition of $\dot{\otimes}^k$-**Sidon** now means that the family of *matrix products* $(\varphi_n(t_1)\cdots\varphi_n(t_k))$ is Sidon on $(T,m)^{\otimes^k}$

Theorem (The union problem)

The union of two "orthogonal" Sidon sets is $\dot{\otimes}^4$*-Sidon*

$$t \mapsto \psi_1(t) \in M_d \quad t \mapsto \psi_2(t) \in M_d$$

$$(\psi_1 \dot{\otimes} \psi_2)(t_1, t_2) = \psi_1(t_1)\psi_2(t_2)$$

Analogous result for Randomly Sidon

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Analogous result for Randomly Sidon

Sidon sets in discrete non-commutative groups

Γ (arbitrary) discrete group, $C^*(\Gamma)$ the full (or maximal) C^*-algebra of Γ i.e. the C^*-algebra generated by the universal representation $U_\Gamma : G \to \mathcal{B}(\mathcal{H})$ of Γ Consider a subset $\Lambda \subset \Gamma$ We set

$$\varphi_t = U_\Gamma(t)$$

Definition

Λ is called is "operator Sidon" if there is a constant C such that for any finitely supported $a : \Lambda \to \mathcal{B}(H)$ (H arbitrary, say $H = \ell_2$) we have

$$\sup_{\substack{z_t \in \mathcal{B}(H) \\ \|z_t\| \le 1}} \{\| \sum\nolimits_{t\in\Lambda} a(t) \otimes z_t \|_{\mathcal{B}(H\otimes_2 H)} \le C \| \sum\nolimits_{t\in\Lambda} a(t) \otimes \varphi_t \|_{\mathcal{B}(H\otimes_2 \mathcal{H})}.$$

Remark: This is much stronger than Sidon but when $\dim(H) = 1$ this reduces to the previous definition of Sidon sets, because

$$\sup_{\substack{z_n \in \mathcal{B}(H) \\ \|z_n\| \le 1}} \{| \sum\nolimits_1^N z_n \otimes a_n| = \sum\nolimits_1^N |a_n|$$

Proposition

Consider $\Lambda \subset \Gamma$. The Following Are Equivalent:

- *(i) Λ is operator Sidon*
- *(ii) $\overline{\mathrm{span}}[\varphi_t \mid t \in \Lambda] \simeq \ell_1(\Lambda)$ completely isomorphically*
- *(iii) $\forall f : \Lambda \to \mathcal{B}(H)$ in $\ell_\infty(\mathcal{B}(H))$ $\exists \tilde{f} : \Gamma \to \mathcal{B}(H)$ of the form*

$$\forall t \in \Gamma \quad \tilde{f}(t) = V^*\pi(t)W$$

for some unitary representation $\pi : \Gamma \to \mathcal{B}(H_\pi)$ and $V, W \in \mathcal{B}(H, H_\pi)$.

Proof is easy:
(i) $\Leftrightarrow$ (ii) is essentially obvious from definitions
proof of (i) $\Rightarrow$ (iii) is by (Arveson) Hahn-Banach:
To any f associate $u_f : \overline{\mathrm{span}}[\varphi_t \mid t \in \Lambda] \to \mathbb{C}$ with $\|u_f\|_{cb} \le C$

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Variants of interpolation pty (iii) were considered for general discrete groups in the 1980's by Bożejko, Picardello and others.

Fundamental Example

$\Gamma = \mathbb{F}_\infty$ with free generators (g_n)

$$\Lambda = \{g_n\}$$

or more generally any free set is operator Sidon

Remark

If $\exists \Lambda \subset \Gamma$ infinite operator Sidon set then Γ is non-amenable, but we do not know whether $\mathbb{F}_2 \subset \Gamma$

Recall Λ is operator Sidon IFF

- (iii) $\forall f : \Lambda \to \mathcal{B}(H)$ in $\ell_\infty(\mathcal{B}(H))$ $\exists F : \Gamma \to \mathcal{B}(H)$ of the form

$$\forall t \in \Gamma \quad F(t) = V^*\pi(t)W$$

for some unitary representation $\pi : \Gamma \to \mathcal{B}(H_\pi)$ and $V, W \in \mathcal{B}(H, H_\pi)$.

Natural operator valued analogue of "Fourier-Stieltjes algebra":

For $F : \Gamma \to \mathcal{B}(H)$

$$\|F\|_{B(\Gamma;\mathcal{B}(H))} = \inf\{\|V\|\|W\| \mid F(\ ) = V^*\pi(\ )W\}$$

Recall when Γ is Abelian, $\widehat{\Gamma}$ is compact then in the case $\mathcal{B}(H) = \mathbb{C}$

$$B(\Gamma) = M(\widehat{\Gamma}) \quad \text{and} \quad \|F\|_{B(\Gamma)} = \|\widehat{F}\|_{M(\widehat{\Gamma})}$$

- (iii) $\forall f \in \ell_\infty(\mathcal{B}(H))$ $\exists F \in B(\Gamma; \mathcal{B}(H))$ such that $F_\Lambda = f$ and

$$\|F\|_{B(\Gamma;\mathcal{B}(H))} \leq C\|f\|_{\ell_\infty(\mathcal{B}(H))}.$$

The following was proved very recently:

Theorem

Operator Sidon sets are stable by union.

Corollary

Finite union of translates of free sets are operator Sidon

Open problem: Is every operator Sidon set the finite union of translates of free sets ?

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The right framework for the preceding is C^*-algebras
Let (φ_n) be a *bounded* sequence in a C^*-algebra

$$A \subset \mathcal{B}(H)$$

Let K be another infinite dimensional Hilbert space (say $K = \ell_2$)
We say that (φ_n) is completely Sidon if there is C such that $\forall N$ and all $a_n \in \mathcal{B}(K)$ we have

$$\sup_{\substack{z_n \in \mathcal{B}(K) \\ \|z_n\| \leq 1}} \{\| \sum\nolimits_1^N z_n \otimes a_n \| \leq C \| \sum\nolimits_1^N a_n \otimes \varphi_n \|.$$

We have also extended to this operator valued setting the result on unions being $\otimes^4$-Sidon...

All the relevant preprints are available on arxiv

Thank you !