Sidon sets in discrete groups

Gilles Pisier
Texas A&M University

Conference in honour of Professor Thangavelu
IISc. Dec. 20, 2017
$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n \in \Lambda} a_n e^{int} \in C(T) \Rightarrow \sum_{n \in \Lambda} |a_n| < \infty$$

**Sidon** sets (and more generally “thin sets” e.g. **Helson** sets) were a very active subject in the 1960’s and 1970’s: Kahane, Malliavin, Varopoulos, Yves Meyer, Bonami + others (in France), Edwards & Gaudry (Australia), Figa-Talamanca (Italy), Rudin, Hewitt & Ross, Rider (USA), Hartman & Ryll-Nardzewski, Bożejko (Poland), Katznelson (Israel), Herz, Drury (Canada)...

The first period culminated with Sam Drury’s solution of “the Union problem”:

Drury (1970): The union of two Sidon sets is again a Sidon set.


Annales Inst. Fourier 2017

Sidon

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n \in \Lambda} a_n e^{int} \in C(T) \Rightarrow \sum_{n \in \Lambda} |a_n| < \infty$$

Equivalently: $\exists C$ such that $\forall A \subset \Lambda$ with $|A| < \infty$

$$\sum_{n \in A} |a_n| \leq C \| \sum_{n \in A} a_n e^{int} \|_{\infty}$$

More generally, let $G$ be a compact Abelian group, $\Lambda = \{ \varphi_n \} \subset \hat{G}$ (characters on $G$), $\Lambda$ is Sidon if $\exists C$ such that $\forall A$ with $|A| < \infty$

$$\sum_{n \in A} |a_n| \leq C \| \sum_{n \in A} a_n \varphi_n \|_{\infty}$$
Fundamental Example

\[ G = T^\mathbb{N} \]

\[ \forall z = (z_n) \in T^\mathbb{N} \quad \varphi_n(z) = z_n \]

\[ \| \sum a_n \varphi_n \|_\infty = \sum |a_n| \quad (C = 1) \]

Note: \((\varphi_n)\) are independent random variables
Hadamard lacunary sequences $n_1 < n_2 < \cdots < n_k, \cdots$ such that

$$\inf_k \frac{n_{k+1}}{n_k} > 1$$

Explicit example

$$n_k = 2^k$$

**Basic Example: Quasi-independent sets**

$\Lambda$ is quasi-independent if all the sums

$$\left\{ \sum_{n \in A} n \mid A \subset \Lambda, |A| < \infty \right\}$$

are distinct numbers

quasi-independent $\Rightarrow$ Sidon

**Main Open Problem**

Is every **Sidon** set a finite union of **quasi-independent** sets?
Bourgain and Lewko (Ann. Inst. Fourier 2017) wondered whether a group environment is needed for the known results about Sidon sets.

**Question**
What remains valid if $\Lambda \subset \hat{G}$ is replaced by a *uniformly bounded* orthonormal system?
Let $\Lambda = \{\varphi_n\} \subset L_\infty(T, m)$ orthonormal in $L_2(T, m)$ ($(T, m)$ any probability space)

(i) We say that $(\varphi_n)$ is Sidon with constant $C$ if for any $N$ and any complex sequence $(a_n)$ we have

$$\sum_1^N |a_n| \leq C \| \sum_1^N a_n \varphi_n \|_\infty.$$ 

(ii) Let $k \geq 1$. We say that $(\varphi_n)$ is $\otimes^k$-Sidon with constant $C$ if the system $\{\varphi_n(t_1) \cdots \varphi_n(t_k)\}$ (or equivalently $\{\varphi_n^{\otimes k}\}$) is Sidon with constant $C$ in $L_\infty(T^k, m^{\otimes k})$.

Crucial remark: For characters on a compact group $T$

$$\text{Sidon} \iff \otimes^k \text{-- Sidon}$$

because

$$\| \sum_1^N a_n \varphi_n \|_\infty = \| \sum_1^N a_n \varphi_n(t_1) \cdots \varphi_n(t_k) \|_{L_\infty(T^k)}$$

Gilles Pisier  
Sidon sets in discrete groups
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(ii) Let \( k \geq 1 \). We say that \((\varphi_n)\) is \( \otimes^k \)-Sidon with constant \(C\) if the system \( \{ \varphi_n(t_1) \cdots \varphi_n(t_k) \} \) (or equivalently \( \{ \varphi_n^{\otimes^k} \} \)) is Sidon with constant \(C\) in \( L_\infty(T^k, m^{\otimes^k}) \).

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\]
The Return of Union Problem

Theorem (Union problem for unif.bded o.n. systems)

Let \((\varphi_n)\) be an orthonormal system bounded in \(L_\infty\). Assume that \((\varphi_n)\) is the union of two (or finitely many) Sidon systems. Then \((\varphi_n)\) is \(\otimes^4\)-Sidon.

But is it Sidon?
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But is it Sidon?
No!

Let \((\varepsilon_n)\) be i.i.d. \(\pm 1\)-valued symmetric random variables (e.g. the Rademacher functions)

Proposition

There are two orthonormal martingale difference sequences \((\varphi_n^+)\) and \((\varphi_n^-)\) in \(L_2\) with \(\text{span}[\varphi_n^+] \perp \text{span}[\varphi_n^-]\) such that

\[(\varphi_n^+) = (\varphi_n^+) = (\varepsilon_n) \text{ in distribution}\]

but their union is not a Sidon system.

More precisely the union of \(\{\varphi_k^+ \mid k \leq n\}\) and \(\{\varphi_k^- \mid k \leq n\}\) has a Sidon constant \(C_n \approx \sqrt{n}\).

Same holds with \((\varepsilon_n)\) replaced by our fundamental example \((z_n)\)
We say that \((\varphi_n)\) is randomly Sidon with constant \(C\) if for any \(N\) and any complex sequence \((a_n)\) we have

\[
\sum_{1}^{N} |a_n| \leq C \text{Average}_{\pm 1} \| \sum_{1}^{N} \pm a_n \varphi_n \|_\infty,
\]

**Theorem**

Let \((\varphi_n)\) be an orthonormal system bounded in \(L_\infty\). The following are equivalent:

- (i) The system \((\varphi_n)\) is randomly Sidon.
- (ii) The system \((\varphi_n)\) is \(\otimes^4\)-Sidon.
- (iii) The system \((\varphi_n)\) is \(\otimes^k\)-Sidon for all \(k \geq 4\).
- (iv) The system \((\varphi_n)\) is \(\otimes^k\)-Sidon for some \(k \geq 4\).

This generalizes Rider’s 1975 result that randomly Sidon implies Sidon for characters.

**Open question:** What about \(k = 2\) or \(k = 3\)?
Two different possibilities have been considered

**EITHER**

(I) Replace the compact Abelian group (e.g. $T$) by a
*non-Abelian compact group* $G$ such as $SO(n), SU(n), U(n), ....$
Then the set $\Lambda \subset \hat{G}$ is a subset of the dual object i.e. the set of
unitary irreducible representations

**OR**

(II) Replace the discrete Abelian group (e.g. $\mathbb{Z}$) by a
*non-Abelian discrete group* $\Gamma$ such as a free group $\mathbb{F}_n$
Then $\Lambda \subset \Gamma$

In both cases I have obtained the analogues of the preceding i.e.
results for general orthonormal functions, that imply the case of
characters as special case using the notion of $\otimes^k$-Sidon
Sidon sets in duals of compact non-commutative groups

$G$ compact non-commutative group
$\hat{G}$ the set of distinct irreps, $d_\pi = \dim(H_\pi)$
$\Lambda \subset \hat{G}$ is called Sidon if $\exists C$ such that $\forall a_\pi \in M_{d_\pi}$ ($\pi \in \Lambda$) we have

$$\sum_{\pi \in \Lambda} d_\pi \text{tr}|a_\pi| \leq C \| \sum_{\pi \in \Lambda} d_\pi \text{tr}(\pi a_\pi) \|_\infty.$$ 

$\Lambda \subset \hat{G}$ is called randomly Sidon if $\exists C$ such that $\forall a_\pi \in M_{d_\pi}$ ($\pi \in \Lambda$) we have

$$\sum_{\pi \in \Lambda} d_\pi \text{tr}|a_\pi| \leq C \mathbb{E} \| \sum_{\pi \in \Lambda} d_\pi \text{tr}(\varepsilon_\pi \pi a_\pi) \|_\infty$$

where $(\varepsilon_\pi)$ are an independent family such that each $\varepsilon_\pi$ is uniformly distributed over $O(d_\pi)$.

**Important Remark (easy proof)** Different randomizations (e.g. Gaussian random matrices) lead to equivalent definitions.
Fundamental example

\[ G = \prod_{n \geq 1} U(d_n) \]

\[ \Lambda = \{ \pi_n \mid n \geq 1 \} \]

\[ \pi_n : G \to U(d_n) \quad n\text{-th coordinate} \]

\[ C = 1 : \sum_{n \geq 1} d_n{\text{tr}}|a_n| = \| \sum_{n \geq 1} d_n{\text{tr}}(\pi_n a_n) \|_\infty. \]

Observe that for the functions \( \varphi_n(i, j) \) defined on \((G, m_G)\) by

\[ \varphi_n(i, j)(g) = \pi_n(g)_{ij} \]

\( \{ d_n^{1/2} \varphi_n(i, j) \mid n \geq 1, 1 \leq i, j \leq d_n \} \) is an orthonormal system.
Rider (1975, unpublished) extended all results previously mentioned to arbitrary compact groups in particular: randomly Sidon implies Sidon (solving the non-commutative union problem) I posted a paper on this on arxiv including (presumably) his proof
Assume given a sequence of finite dimensions $d_n$.
For each $n$ let $(\varphi_n)$ be a random matrix of size $d_n \times d_n$ on $(T, m)$. We call this a "matricial system":

$$\varphi_n = [\varphi_n(i, j)]$$

or rather for $t \in T$

$$\varphi_n(t) = [\varphi_n(i, j)(t)]$$
The **uniform boundedness condition** becomes

\[ \exists C' \, \forall n \, \| \varphi_n \|_{L^\infty(M_{d_n})} \leq C'. \]

The **orthonormality condition** becomes:

\[ \{ d_n^{1/2} \varphi_n(i,j) \mid n \geq 1, 1 \leq i, j \leq d_n \} \]

is an orthonormal system.
The definition of $\hat{\otimes}^k$-Sidon now means that the family of matrix products $(\varphi_n(t_1) \cdots \varphi_n(t_k))$ is Sidon on $(T, m) \otimes^k$

**Theorem (The union problem)**

*The union of two “orthogonal” Sidon sets is $\hat{\otimes}^4$-Sidon*

\[ t \mapsto \psi_1(t) \in M_d \quad t \mapsto \psi_2(t) \in M_d \]

\[ (\psi_1 \hat{\otimes} \psi_2)(t_1, t_2) = \psi_1(t_1)\psi_2(t_2) \]

Analogous result for Randomly Sidon
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**Theorem (The union problem)**

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Analogous result for Randomly Sidon
Γ (arbitrary) discrete group,  \( C^*(\Gamma) \) the full (or maximal) \( C^* \)-algebra of \( \Gamma \) i.e. the \( C^* \)-algebra generated by the universal representation \( U_\Gamma : G \to B(\mathcal{H}) \) of \( \Gamma \) Consider a subset \( \Lambda \subset \Gamma \) We set

\[
\varphi_t = U_\Gamma(t)
\]

**Definition**

\( \Lambda \) is called is “operator Sidon” if there is a constant \( C \) such that for any finitely supported \( a : \Lambda \to B(H) \) (\( H \) arbitrary, say \( H = \ell_2 \)) we have

\[
\sup_{z_t \in B(H) \atop \|z_t\| \leq 1} \left\{ \left\| \sum_{t \in \Lambda} a(t) \otimes z_t \right\|_{B(H \otimes_2 H)} \right\} \leq C \left\| \sum_{t \in \Lambda} a(t) \otimes \varphi_t \right\|_{B(H \otimes_2 H)}.
\]

**Remark:** This is much stronger than Sidon but when \( \dim(H) = 1 \) this reduces to the previous definition of Sidon sets, because

\[
\sup_{z_n \in B(H) \atop \|z_n\| \leq 1} \left\{ \left| \sum_{1}^{N} z_n \otimes a_n \right| = \sum_{1}^{N} |a_n| \right\}
\]
Proposition

Consider $\Lambda \subset \Gamma$. The Following Are Equivalent:

- (i) $\Lambda$ is operator Sidon
- (ii) $\overline{\text{span}}[\varphi_t | t \in \Lambda] \simeq \ell_1(\Lambda)$ completely isomorphically
- (iii) $\forall f : \Lambda \to \mathcal{B}(H)$ in $\ell_\infty(\mathcal{B}(H))$ $\exists \tilde{f} : \Gamma \to \mathcal{B}(H)$ of the form

\[
\forall t \in \Gamma \quad \tilde{f}(t) = V^* \pi(t)W
\]

for some unitary representation $\pi : \Gamma \to \mathcal{B}(H_\pi)$ and $V, W \in \mathcal{B}(H, H_\pi)$.

Proof is easy:

(i) $\iff$ (ii) is essentially obvious from definitions

proof of (i) $\implies$ (iii) is by (Arveson) Hahn-Banach:

To any $f$ associate $u_f : \overline{\text{span}}[\varphi_t | t \in \Lambda] \to \mathbb{C}$ with $\|u_f\|_{cb} \leq C$

Variants of interpolation pty (iii) were considered for general discrete groups in the 1980’s by Bożejko, Picardello and others.
\[ \Gamma = \mathbb{IF}_\infty \text{ with free generators } (g_n) \]

\[ \Lambda = \{g_n\} \]

or more generally any free set is operator Sidon

Remark

If \( \exists \Lambda \subset \Gamma \) infinite operator Sidon set then \( \Gamma \) is non-amenable, but we do not know whether \( \mathbb{IF}_2 \subset \Gamma \)
Recall $\Lambda$ is operator Sidon IFF

- (iii) $\forall f : \Lambda \rightarrow \mathcal{B}(H)$ in $\ell_\infty(\mathcal{B}(H))$ $\exists F : \Gamma \rightarrow \mathcal{B}(H)$ of the form

$$\forall t \in \Gamma \quad F(t) = V^*\pi(t)W$$

for some unitary representation $\pi : \Gamma \rightarrow \mathcal{B}(H_\pi)$ and $V, W \in \mathcal{B}(H, H_\pi)$.

**Natural operator valued analogue of “Fourier-Stieltjes algebra”:**

For $F : \Gamma \rightarrow \mathcal{B}(H)$

$$\|F\|_{\mathcal{B}(\Gamma; \mathcal{B}(H))} = \inf \{ \|V\|\|W\| : F(\ ) = V^*\pi(\ )W \}$$

Recall when $\Gamma$ is Abelian, $\hat{\Gamma}$ is compact then in the case $\mathcal{B}(H) = \mathbb{C}$

$$\mathcal{B}(\Gamma) = M(\hat{\Gamma}) \quad \text{and} \quad \|F\|_{\mathcal{B}(\Gamma)} = \|\hat{F}\|_{M(\hat{\Gamma})}$$

- (iii) $\forall f \in \ell_\infty(\mathcal{B}(H))$ $\exists F \in \mathcal{B}(\Gamma; \mathcal{B}(H))$ such that $F_\Lambda = f$ and

$$\|F\|_{\mathcal{B}(\Gamma; \mathcal{B}(H))} \leq C\|f\|_{\ell_\infty(\mathcal{B}(H))}.$$
The following was proved very recently:

**Theorem**

*Operator Sidon sets are stable by union.*

**Corollary**

*Finite union of translates of free sets are operator Sidon*

**Open problem:** Is every operator Sidon set the finite union of translates of free sets?
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*Finite union of translates of free sets are operator Sidon*

**Open problem:** Is every operator Sidon set the finite union of translates of free sets?
The right framework for the preceding is $C^*$-algebras
Let $(\varphi_n)$ be a bounded sequence in a $C^*$-algebra

$$A \subset \mathcal{B}(H)$$

Let $K$ be another infinite dimensional Hilbert space (say $K = \ell_2$)
We say that $(\varphi_n)$ is completely Sidon if there is $C$ such that $\forall N$ and all $a_n \in \mathcal{B}(K)$ we have

$$\sup_{\|z_n\| \leq 1} \left\{ \| \sum_{1}^{N} z_n \otimes a_n \| \leq C \| \sum_{1}^{N} a_n \otimes \varphi_n \| \right.$$ 

We have also extended to this operator valued setting the result on unions being $\otimes^4$-Sidon...
All the relevant preprints are available on arxiv

Thank you!