Foundations of Cooperative Game Theory

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Important Note

- The contents of this presentation are (mostly) taken from the following text book:
  Y. Narahari. *Game Theory and Mechanism Design*. IISc Press and World Scientific, 2014.

- *Take Away From this Presentation* is certain important *Foundational Concepts from Cooperative Game Theory*. This presentation by no means does not cover all foundational concepts as well as does not cover advanced topics in cooperative game theory.

Outline of the Talk

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Basic Definitions

- **Definition:** A cooperative game with transferable utility is defined as the pair (N, v) where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \rightarrow \mathbb{R}$ is a characteristic function, with $v(\phi) = 0$.

- The above game is also known as:
  - Coalitional Game,
  - Game in Characteristic Form,
  - TU Game, etc.

Examples of TU Games

- **Example 1 [Divide the Dollar Game]:** Consider that three players wish to divide a total wealth of 300 among themselves. Each player can propose a payoff such that no player's payoff is negative and the sum of all the payoffs does not exceed 300. Assume that the players get a zero payoff unless all three players propose the same allocation. The corresponding cooperative game is:
  - $N = \{1, 2, 3\}$
  - $v(\{1\}) = 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0,$
    $v(\{1, 2\}) = 0, \quad v(\{1, 3\}) = 0, \quad v(\{2, 3\}) = 0,$
    $v(\{1, 2, 3\}) = 300$

Examples of TU Games (Cont.)

- **Example 2 [A Voting Game]:** Consider that the Parliament of a certain Nation has four political parties $1, 2, 3, 4$ with $45, 25, 15, 12$ members respectively. To pass any bill, at least 51 votes are required. This situation could be modeled as:
  - $N = \{1, 2, 3, 4\}$
  - $v(\{1\}) = 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0,$
    $v(\{12\}) = v(\{13\}) = v(\{14\}) = v(\{123\}) = v(\{124\}) = v(\{134\}) =$
    $v(\{234\}) = v(\{1234\}) = 1,$
    $v(\{23\}) = v(\{24\}) = v(\{34\}) = 0$

Outline of the Talk

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TU Games with Special Structures

- **Monotonic Game:** A TU game (N, v) is called *monotonic* if

$$v(C) \leq v(D), \ \ \forall C \subseteq D \subseteq N.$$

- **Superadditive Game:** A TU game (N, v) is called *superadditive* if $\forall C, D \subseteq N$ such that $C \cap D = \phi$:

$$v(C \cup D) \geq v(C) + v(D).$$

Examples of Superadditive Games

- **Example 1:** Consider the Voting game.
  - $N = \{1, 2, 3\}$
  - $v(\{1\}) = 10, \quad v(\{2\}) = 15, \quad v(\{3\}) = 20,$
    $v(\{12\}) = 20, \quad v(\{13\}) = 30, \quad v(\{23\}) = 35, \quad v(\{123\}) = 40.$

  It is NOT a superadditive game (for instance, $v(123) < v(13) + v(2)$).

- **Example 2:** Consider a TU game as follows: $N = 1, 2, 3, 4$ and $v(\cdot)$ is given by $v(1) = v(2) = v(3) = 0$ and
  $v(12) = v(13) = v(23) = v(123) = 300.$
  This is a superadditive TU game.

Essential Superadditive Games

- **Essential Superadditive Game:** A superadditive game (N, v) is said to be *inessential* if

$$\sum_{i \in N} v(i) = v(N).$$

and *essential* otherwise.

- If (N, v) is inessential, then:

$$\sum_{i \in C} v(i) = v(C), \;\; \forall C \subseteq N.$$

Question: Proof?

Strategic Equivalence of TU Games

- Two TU games (N, v) and (N, w) are said to be strategically equivalent if there exist constants $c_1, c_2, \ldots, c_n$ and $b > 0$ such that

$$w(C) = b(v(C) + \sum_{i \in C} c_i), \ \forall C \subseteq N.$$

- Intuitively, strategic equivalence means that the *dynamics* among the players would be identical in the two games.
- Any essential superadditive n-person TU game is strategically equivalent to a unique game wherein

$$N = \{1, 2, \ldots, n\}$$

$$v(1) = v(2) = \ldots = v(n) = 0; v(N) = 1$$

$$0 \leq v(C) \leq 1, \ \ \forall C \subseteq N.$$

This game is called the *0-1 normalization* of the original TU game.

Method to Convert to 0-1 Normalization Form

- Any essential superadditive (N, v) can be converted to $(0, 1)$ normal form by using: $\forall S \subseteq N$,

$$u(S) = \frac{v(S) - \sum_{i \in S} v(i)}{v(N) - \sum_{i \in N} v(i)}.$$

- All $(0, 1)$-normalized games are essential. (*Question:* Proof?)

Convex Games

- A TU game is said to be convex if

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D), \ \ \forall C, D \subseteq N.$$

- Suppose C and D are disjoint. Then the above definition becomes:

$$v(C \cup D) \geq v(C) + v(D), \ \ \forall C, D \subseteq N \ \ C \cap D = \phi.$$

- Hence, every convex game is superadditive. However, the converse need not be true.
- Another equivalent definition for convex games is:

$$v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D), \ \ \forall C \subseteq D \subseteq N, \ \forall i \in N \setminus D.$$

Outline of the Talk

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Payoff Allocations and Solution Concepts

- Recall that a cooperative game (N, v) is said to be super-additive if for any two coalitions A and B such that $A \cap B = \Phi$ and $v(A \cup B) \geq v(A) + v(B)$.
- In super-additive cooperative games, grand coalition forms.
- **Key Question:** How should the grand coalition that forms divide its winnings among its members?
- Solution Concepts: (i) Core, (ii) Shapley Value, (iii) Stable Sets, (iv) Bargaining Set, (v) Kernal, (vi) Nucleolus, etc.

Imputations

- A payoff allocation $x = (x_1, x_2, \ldots, x_n)$ is any vector in $\mathbb{R}^n$ where x_i is the utility payoff to player i
- Any allocation $x = (x_1, x_2, \ldots, x_n)$ is said to be *feasible* for a coalition C if and only if
$$\sum_{i \in C} x_i \leq v(C)$$
- Any allocation $x = (x_1, x_2, \ldots, x_n)$ is said to be *individually rational*, if $x_i \geq v(\{i\})$, $\forall i \in N$
- Any allocation $x = (x_1, x_2, \ldots, x_n)$ is said to be *collectively rational*, if $\sum_{i \in N} x_i = v(N)$, $\forall i \in N$

Imputations (Cont.)

- **Imputation:** Given a TU game (N, v), an imputation is a payoff allocation that satisfies *individual rationality* and *collective rationality*.
- **Domination of Imputation:** An imputation $x = (x_1, x_2, \ldots, x_n)$ of a TU game (N, v) is said to dominate an imputation $y = (y_1, y_2, \ldots, y_n)$ if there exists a coalition C such that

$$\sum_{i \in C} x_i \leq v(C); \quad \textit{and} \quad x_i > y_i \;\; \forall i \in C.$$

- Any allocation $x = (x_1, x_2, \ldots, x_n)$ is said to be *coalitionally rational*, if $\sum_{i \in C} x_i \geq v(C)$, $\forall C \subseteq N$.

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The Core

- The core of a TU game (N, v) is the set of all payoff allocations that are individually rational, coalitionally rational, and collectively rational. That is,

$$Core = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N); \sum_{i \in C} x_i \geq v(C), \forall C \subseteq N\}$$

- We say a coalition C can improve on (or blocks) an allocation $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ if

$$v(C) > \sum_{i \in C} x_i.$$

- The core of (N, v) is the set of allocations x such that x is feasible for N and no coalition $C \subseteq N$ can improve upon it.

The Core (Cont.)

- **Example 1 [Divide the Dollar Game]:** Consider that three players wish to divide a total wealth of 300 among themselves. Each player can propose a payoff such that no player's payoff is negative and the sum of all the payoffs does not exceed 300. Assume that the players get a zero payoff unless all three players propose the same allocation. The corresponding cooperative game is:
  - $N = \{1, 2, 3\}$
  - $v(\{1\}) = 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0,$
    $v(\{1,2\}) = 0, \quad v(\{1,3\}) = 0, \quad v(\{2,3\}) = 0,$
    $v(\{1,2,3\}) = 300$

- For the above game, the core is given by:

$$\{(x_1, x_2, x_3) \in \mathbb{R}^n \ : \ x_1 + x_2 + x_3 = 300, \ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0\}.$$

The Core (Cont.)

- **Example 2 [Glove Market]:** Consider a glove market. Let $N = \{1, 2, 3, 4, 5\}$. The first two players can supply left gloves and the other three players can supply right gloves; $N_L = \{1, 2\}$ and $N_R = \{3, 4, 5\}$. Suppose the worth of each coalition is the number of matched pairs that it can assemble. That is,

$$v(C) = min\{|C \cap N_L|, |C \cap N_R\}$$

  The core for this game is a singleton set $\{(1, 1, 0, 0, 0)\}$.
- On the other hand, if $N_L = \{1, 2, 3\}$ and $N_R = \{4, 5\}$, then the core of the game would be $\{(0, 0, 0, 1, 1)\}$.
- If the cardinalities of the two sets N_L and N_R are equal, then the core of the game would be $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$.

The Core of Convex Games

- **Theorem:** Given a convex game (N, v), the allocation $x = (x_1, \ldots, x_n)$ defined by

$$x_1 = v(\{1\})$$
$$x_2 = v(\{1,2\}) - v(\{1\})$$
$$.....$$
$$x + n = v(\{1,2,\ldots,n\}) - v(\{1,2,\ldots,n-1\})$$

  belong to the core of the game.
- The above result holds for any allocation obtained through a permutation of the set N.
- Let $\Pi(N)$ denote the set of all permutations of the $N = \{1, 2, \ldots, n\}$. Suppose $\pi \in \Pi(N)$ is any permutation.
- Let $P(\pi, i)$ denote the set of players who are predecessors of i in the permutation π. That is, $P(\pi, i) = \{j \in N \ : \ \pi(j) < \pi(i)\}$.

The Core of Convex Games

- The marginal contribution of player i to his predecessors in π:

$$m(\pi, i) = v(P(\pi, i) \cup \{i\}) - v(P(\pi, i)).$$

- Suppose (N, v) is a convex game and $\pi \in \Phi(N)$ is any permutation. Then the allocation $(m(\pi, 1), m(\pi, 2), \ldots, m(\pi, n))$ belongs to the core of (N, v).
- The core of a convex game is a convex set.
- The number of elements in the Core of a TU game can be empty, singleton, or more.

Outline of the Talk

1. Basic Definitions
2. TU Games with Special Structures
3. Imputations
4. The Core
5. The Shapley Value
6. Stable Sets
7. Concluding Remarks

Outline of the Talk

The Shapley Value

- Shapley value is a solution concept which is motivated by the need to have a theory that would predict a unique expected payoff allocation for every given coalitional game
- The Shapley value concept was proposed by Shapley in 1953, following an axiomatic approach. This was part of his doctoral dissertation at the Princeton University. Given a cooperative game (N, v), the Shapley value is denoted by $\phi(v)$:

$$\phi(v) = \{\phi_i(v), \phi_2(v), \ldots, \phi_n(v)\}$$

  where $\phi_i(v)$ is the expected payoff to player i
- Shapley proposed three axioms: Symmetry, Linearity, and Carrier

The Shapley Value (Cont.)

- Let (N, v) be a coalitional game and π be a permutation of the players in N. Let $(N, \pi v)$ be a coalitional game such that

$$\pi v(\{\pi(i) : i \in C\}) = v(C), \forall C \subseteq N$$

  That is, the role of any player i in (N, v) is same as the role of player $\pi(i)$ in $(N, \pi v)$

- *Symmetry:* For any $v \in \mathbb{R}^{2^n - 1}$, any permutation π on N, and any player $i \in N$,

$$\phi_{\pi(i)}(\pi v) = \phi_i(v)$$

- *Linearity:* Let (N, v) and (N, w) be any two coalitional games. Suppose $p \in [0, 1]$. Define the game $(N, pv + (1 - p)w)$ as follows:

$$(pv + (1 - p)w)(C) = pv(C) + (1 - p)w(C), \ \forall C \subseteq N$$

  Then the axiom of linearity says that

$$\phi_i(pv + (1 - p)w) = p\phi_i(v) + (1 - p)\phi_i(w)$$

The Shapley Value (Cont.)

- *Carrier:* A coalition D is said to ba a carrier of a coalitional game (N, v) if
$$v(C) = v(C \cap D), \ \forall C \subseteq N$$
The carrier axiom states that, for any (N, v) and any carrier D,
$$\sum_{i \in D} \phi_i(v) = v(D) = v(N)$$

The Shapley's Theorem

- **Theorem:** There is exactly one mapping $\phi : \mathbb{R}^{2^N - 1} \to \mathbb{R}^N$ that satisfies Symmetry, Linearity, and Carrier axioms. This function satisfies: $\forall i \in N$, $\forall v \in \mathbb{R}^{2^N - 1}$,

$$\phi_i(v) = \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(n - |C| - 1)!}{n!} \{v(C \cup \{i\}) - v(C)\}$$

The Shapley's Theorem: An Illustration

- **Example:** Consider the following cooperative game:
  - $N = \{1, 2, 3\}$;
  - $v(1) = v(2) = v(3) = v(23) = 0$, $v(12) = v(13) = v(123) = 300$

- Then we have that

$$\phi_1(v) = \frac{2}{6}v(1) + \frac{1}{6}(v(12) - v(2)) + \frac{1}{6}(v(13) - v(3)) + \frac{2}{6}(v(123) - v(23))$$

- It can be easily computed that
  - $\phi_1(v) = 200$,
  - $\phi_2(v) = 50$, and
  - $\phi_3(v) = 50$

Alternative Formulae for Shapley Value

- Let $C \subseteq N$ and $i \notin C$. The marginal contribution of i to C, denoted by $m(C, i)$, is given by:

$$m(C, i) = v(C \cup \{i\}) - v(C).$$

- Given any permutation $\pi \in \Pi(N)$, the set of predecessors $P(\pi, i)$ of i in the permutation π is given by:

$$P(\pi, i) = \{j : \pi(j) < \pi(i)\}.$$

- The Shapley value, $\phi_i(v)$ is the average marginal contribution of i to any coalition of N assuming all orderings are equally likely. That is,

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m(P(\pi, i), i), \; \forall i \in N.$$

Shapley Value of Convex Games

- **Theorem:** Let (N, v) be a convex game. Then $\phi(v)$ belong the core.
- Let $\Pi(N)$ be the collection of all permutations of the set N. Suppose that $\pi \in \Pi(N)$ is any permutation.
- Define the allocation $y^\pi \in \mathbb{R}^n$ as:

$$y^\pi = (m(\pi, 1), m(\pi, 2), \ldots, m(\pi, n)).$$

- It is known that y^π belongs to the Core of convex game.
- We also know that the Shapley value i-th player:

$$\phi_i = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m(P(\pi, i), i).$$

Shapley Value of Convex Games (Cont.)

- Then we have that

$$\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$$

$$\phi(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} y^{\pi}.$$

- That is, $\phi(v)$ is a convex combination of all vectors y^{π}; $\pi \in \Pi(N)$.
- Since the core is a convex set, it follows immediately that $\phi(v)$ belongs the Core.

Outline of the Talk

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Excess

- Given a TU game (N, v), a coalition C, and an allocation $x = (x_1, \ldots, x_n)$, the excess of C at x is defined as

$$e(C, x) = v(C) - \sum_{i \in C} x_i$$

- **Domination:** An imputation $x = (x_1, \ldots, x_n)$ is said to dominate another imputation $y = (y_1, \ldots, y_n)$ if there is some coalition $C \subseteq N$ such that (i) $e(C, x) \geq 0$, and (ii) $x_i > y_i$, $\forall i \in C$.
- **Undominated Imputation:** An imputation $x = (x_1, \ldots, x_n)$ is said to be undominated if no other imputation dominates it.

Internal and External Stability

- **Internal Stability:** Given a TU game (N, v), a set of imputations Z is said to be internally stable if

$$\forall x, y \in Z,\ \forall C \subseteq N,\ x_i > y_i,\ \forall i \in C \Rightarrow e(C, x) < 0.$$

- **External Stability:** Given a TU game (N, v), a set of imputations Z is said to be externally stable if every imputation not in Z is dominated by some imputation in Z.

- **Stable Set:** A stable set of a TU game (N, v) is a set of imputations Z satisfying internal stability as well as external stability.

Outline of the Talk

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Summary of the Presentation

This presentation covered the following aspects:

- Introduction to TU Games with examples,
- TU games with special structures,
- Imputations,
- Solution concepts such as Core, Shapley value, Nucleolus, etc.

Some Important Text Books

- Georgios Chalkiadakis, Edith Elkind, and Michael Wooldridge. *Computational aspects of cooperative game theory*. Morgan & Claypool, 2011.
- Bezalel Peleg and Peter Sudhlter. *Introduction to the theory of cooperative cooperative games*. Springer, 2nd edition, 2007.
- Y. Narahari. *Game Theory and Mechanism Design*. IISc Press and World Scientific, 2014.
- P. D. Straffin. *Game Theory and Strategy.* New Mathematics Library, 1993.
- R. Branzei, D. Dimitrov, S. Tijs. *Models in Cooperative Game Theory*. Springer 2008.

THANK YOU