# Anti-Coordination Games and Graph Colouring

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- The minimum possible value of $k$ such that there is a proper-k-colouring is called the *chromatic number* of the graph

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- The utility of agent $i$ agains an agent $j$, in a bilateral game, is given by $\pi(s_i, s_j)$.
- A crucial assumption is that every player chooses the same action in all bilateral games.

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- Many applications involving negative interactions.
- Negative interactions are modelled using anti-coordination games.

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- when there is a kind of predation of one strategy on the other; e.g., Hawk-Dove game and Chicken game.

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- The bilateral game is anti-coordination game. It means that the pure strategy equilibria are $(A, B)$ and $(B, A)$.
- This is equivalent to saying

$$\pi(B, A) > \pi(A, A);\ \pi(AB) > \pi(B, B)$$

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- The bilateral game has a unique mixed equilibrium in which the probability of playing $A$ is

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- A profile $s$ is a Nash equilibrium of the social game if it satisfies

$$\forall i,\ \forall s_i', \quad \pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}).$$

# Bramoullé's Model

**Theorem**

*A profile s is a Nash equilibrium if and only if for every agent i,*

$$n_{i,A} < p_A n_i \implies s_i = A \text{ and } n_{i,A} < p_A n_i \implies s_i = B.$$

*Here $n_i$ refers to the number of neighbours of $i$; $n_{i,A}$ refers to the number of neighbours of $i$ playing $A$.*

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- Many results from Potential games can be applied.

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**Theorem**

*A graph is bipartite if and only if there exists $s, \pi_A, \pi_B$ such that $\phi(s, \pi_A, \pi_B, g) = 0$.*

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- The utility to the player is the expected payoff he receives in these random interactions.

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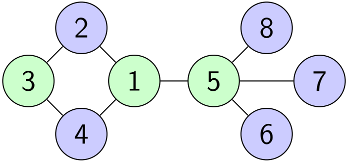
# Problem with Bramoullé's Model

Consider the network with 8 agents and the two configurations.



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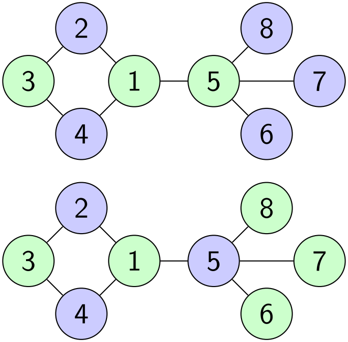


Both the configurations are Nash equilibrium. Note that the graph is bipartite (see the second configuration). However, the first configurations is not a proper colouring. Thus the Bramoullé's model does not capture the anti-coordination in a stict sense.

# Game Theoretic View: Recent Studies

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- Mainly these works analyse the greedy algorithm. Each time, an agent picks a colour not used by the neighbours.
- It is proved that this greedy algorithm convergences to a proper colouring. The probability of convergence is not 1.
- The model is essentially same as the model by by Bramoullé. Also, Bramoullé's model assumes only two choices for the agents.

# Our Model

- The utility function is given by

$$\pi_i(s) = \underbrace{-\sum_{j \in N_i} \mathbb{1}_{s_i = s_j}}_{\text{Term1}} + \underbrace{\frac{1}{K_i} \sum_{k,j \in N_i} \mathbb{1}_{s_k = s_j}}_{\text{Term2}}, \qquad (1)$$

  where

$$K_i = 2\binom{|N_i|}{2}$$

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- The second term counts the number of neighbours having same colour and thus represents the benefit of having minimum number of colours in the neighbourhood.

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- Thus Term 1 will be higher if no neighbour of player $i$ has same colour as the player $i$.
- In other words, proper colouring will always be a Nash equilibrium.
- In fact, we have the following result: A pure strategy is a Nash equilibrium if and only if it is proper colouring.

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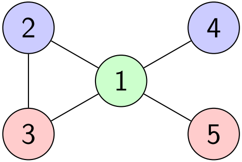

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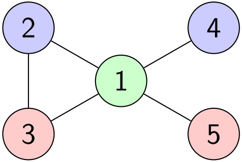

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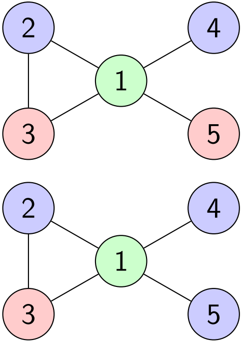

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- Given a proper colouring, for a player $i$ and her neighbourhood $N(i)$, define the **neighbourhood conflict count (NCC)** of player $i$ as the number of pairs of agents belonging to $N(i)$ that have different colours.

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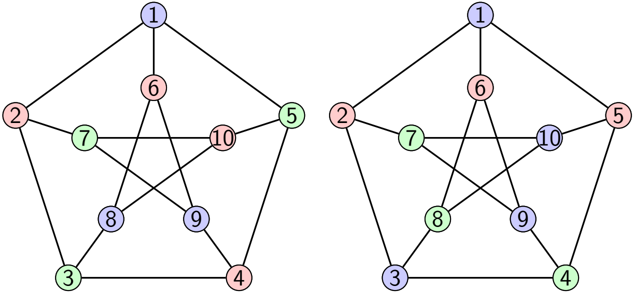
- Each such pair of agents in the neighbourhood of $i$ whose colours are different, is termed as **a neighbourhood conflict of player $i$**.
- Pareto equilibria correspond to minal "neighborhood conflicting" profiles.

# Our Model and Results

- Pareto equilibria need not be unique.



# Question

How do we obtain minimal colouring?

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- The game has too many Nash equilibrium.
- The game is not Potential.
- Note that our game is not a local interaction game (in the sense of Blume). It should be understood as a game with networked agents.
- To get the minimal colouring, we consider a modification of the game.

# Modified Game

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- This requires a 2-hop neighbourhood information.

# Modified Game: Main Result

Theorem

*Every Nash equilibrium is a Pareto and hence it is minimal.*

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- Hence the best response dynamics gives minimal colouring.

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- He picks the colour which is picked by most of $j$'s neighbours.

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- So, the algorithm converges.
- In fact, the algorithm will reach the steady state in finite time with probability 1.

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- The learning scheme works irrespective of the number of colours.
- We can handle general anti-coordination games.
- There is no clear definition for anti-coordination games with many players. Graph colouring is one way of defining anti-coordination game.
- The idea of the modified game can help in studying socially optimal equilibrium in general games.

# Some References


- Blume, The statistical mechanics of strategic interaction, Games and Economic Behavior, 1993.

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Questions, Comments?

Thank You