Leader-follower and coupled-constraint games

Ankur A. Kulkarni

Systems and Control Engineering
Indian Institute of Technology Bombay

Notation

- $\mathcal{N} = \{1, \cdots, N\}$ set of players.
- Player chooses action x_i; $x \triangleq (x_1, \cdots, x_N)$ and $x^{-i} \triangleq (x_1, \cdots, x_{i-1}, x_{i+1}, x_N)$
- Loss function $\varphi_i(x)$ which each player seeks to minimize
- Suppose player $i \in \mathcal{N} = \{1, \cdots, N\}$ solves

$$\begin{array}{ll} \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{-i}) \\ \text{subject to} & x_i \in X_i, \end{array}$$

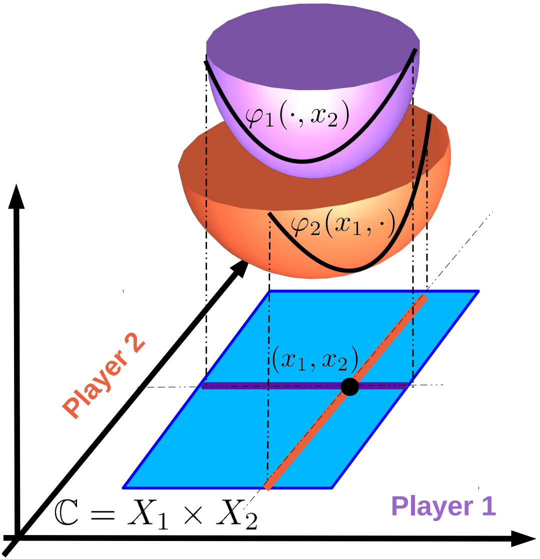
where X_i is a closed and convex set in $\mathbb{R}^{m_i}$.

$$X \triangleq \prod_{i \in \mathcal{N}} X_i.$$

- For each $i \in \mathcal{N}$, $\varphi_i(\cdot, x^{-i})$ is assumed to be convex for each x^{-i}.
- **Nash equilibrium:** $x^* \in X$ such that

$$x_i^* \in \arg\min_{x_i \in X_i} \varphi_i(x_i, x^{*,-i})$$

Game (classical)



Best response and Kakutani's fixed point theorem

- Best response

$$\mathcal{R}_i(x^{-i}) = \arg\min_{x_i \in X_i} \varphi_i(x_i; x^{-i})$$

$$\mathcal{R}(x) = \prod_{i \in \mathcal{N}} \mathcal{R}_i(x^{-i})$$

- $\mathcal{R}$ maps X to subsets of X
- $x^* = (x_1^*, \cdots, x_N^*)$ is a Nash equilibrium if and only if

$$x^* \in \mathcal{R}(x^*),$$

i.e., if $x^* \in \mathrm{Fix}(\mathcal{R})$.

Theorem (Kakutani)

Let $X \subseteq \mathbb{R}^n$ and let $T : X \to 2^X$. If

- *X is convex and compact*
- *T is convex-valued*
- *T has closed graph (i.e., $\{(x, y) | y \in T(x), x \in X\}$ is closed) (equivalently, T is upper semi-continuous)*

then $\mathrm{Fix}(T) \neq \emptyset$.

Coupled constraints

- In the classical setting, actions available to a player are not constrained by the actions of other players
- Generalizing this: suppose we have a set $\mathbb{C} \subseteq \mathbb{R}^m = \mathbb{R}^{\sum m_i}$ so that players are constrained to choose their actions such that

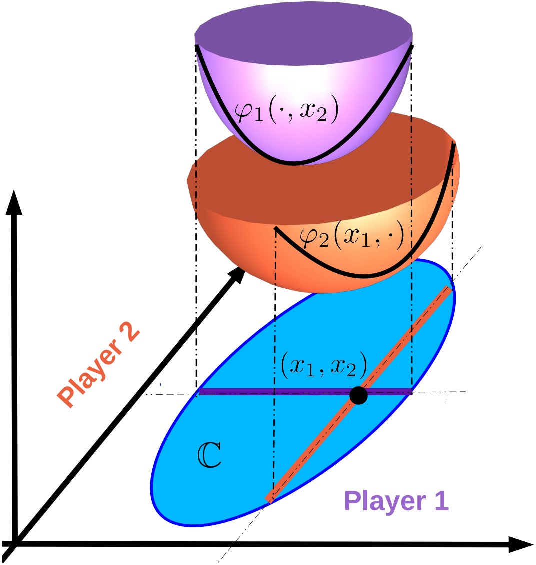
$$(x_1, \cdots, x_N) \in \mathbb{C}.$$

- In other words the *rectangular* set $X \subseteq \mathbb{R}^m$ is now replaced by a general set $\mathbb{C}$.
- **Examples:** board games, capacity constraints, physical laws, etc
- **Nash equilibrium:** $x^* = (x_1^*, \cdots, x_N^*)$ such that x_i^* solves

$$\mathrm{A}_i(x^{*,-i}) \qquad \begin{array}{ll} \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{*,-i}) \\ \text{subject to} & (x_i, x^{*,-i}) \in \mathbb{C}. \end{array}$$

- Actions available to a player are now a *function* of the actions of other players
- Notice that $\mathbb{C}$ does not depend on i; thus it is a common binding constraint for all players

Shared-constraint game



Challenges

- Meaningfulness: how does one make sense of a *simultaneous* move game with coupled constraints?
- Existence: What happens to Kakutani's fixed point theorem?
- Best response:

$$\mathcal{R}_i(x^{-i}) = \arg \min_{x_i \in K_i(x^{-i})} \varphi_i(x; x^{-i})$$

  where $K_i(x^{-i}) \triangleq \{x_i | (x_i, x^{-i}) \in \mathbb{C}\}$.
- Kakutani requires $\mathcal{R}_i$ to be upper semi-continuous, for which *continuity* of K_i is as good as necessary
- Unfortunately this does not hold in general

Definitions of upper and lower semicontinuity

- **Upper:** If $x_n \to x$ and $y_n \in T(x_n)$ such that $y_n \to y$ then $y \in T(x)$.
- **Lower:** If $x_n \to x$ and $y \in T(x)$ then there exists $y_n \in T(x_n)$ for all n such that $y_n \to y$.
- **Continuity:** Upper + Lower

Rosen's argument

- Define

$$\Psi(y,x) = \sum_{i\in\mathcal{N}} \varphi_i(y_i; x^{-i})$$

- Notice that for the classical game,

$$\mathcal{R}(x) = \arg\min_{y\in X} \Psi(y,x)$$

Hence $x^* \in \mathcal{R}(x^*)$ if and only if $x^* \in \arg\min_{y\in X} \Psi(y,x^*)$

- What about the game with a coupled shared constraint?
- Rosen shows that one direction is true with X replaced by $\mathbb{C}$!

$$\text{If } x^* \in \arg\min_{y\in\mathbb{C}} \Psi(y,x^*) \implies x^* \in \mathcal{R}(x^*).$$

- Thus it suffices to look for a fixed point of the new map $\Upsilon : \mathbb{C} \to 2^{\mathbb{C}}$ where

$$\Upsilon(x) = \arg\min_{y\in\mathbb{C}} \Psi(y,x).$$

- Observe that the constraints in the minimization *do not* depend on x. Hence Kakutani can be applied directly with the same assumptions as before.

More about Rosen's argument (1965) [Rosen, 1965]

Nash equilibrium

A strategy tuple $x \equiv (x_1, x_2, \cdots, x_N)$ where $x_i \in \mathrm{SOL}(A_i(x^{-i}))$ for all $i \in \mathcal{N}$.

$$\Updownarrow \qquad\qquad\qquad\qquad \Uparrow \quad \not\Downarrow$$

$$x \in \mathcal{R}(x) \qquad\qquad\qquad\qquad x \in \Upsilon(x)$$

$$\mathcal{R}(x) := \arg \inf_{u \in K(x)} \sum_{i \in \mathcal{N}} \varphi_i(u_i, x^{-i}) \qquad \Upsilon(x) := \arg \inf_{u \in \mathbb{C}} \sum_{i \in \mathcal{N}} \varphi_i(u_i, x^{-i})$$

- $x \in \mathcal{R}(x)$: intractable, $x \in \Upsilon(x)$: tractable
- Rosen shows that $\mathrm{Fix}(\Upsilon) \subseteq \mathrm{Fix}(\mathcal{R})$. What more can we say about $\mathrm{Fix}(\Upsilon)$? Rosen calls these the *normalized* Nash equilibria. We will see more about them soon.
- What about the reverse inclusion? Is that ever true?
- And what about the cases where Kakutani does not apply due to breakdown of other assumptions:
  - Compactness of $\mathbb{C}$
  - Convex-valuedness of Φ
  - Convexity of $\mathbb{C}$
- We will address this as well..

VIs, QVIs

Let $F(x) = \left(\nabla_1 \varphi_1(x) \quad \cdots \quad \nabla_N \varphi_N(x)\right).$

A NE solves the **quasi-variational inequality** (QVI)

$$\text{Find } x \in K(x) \text{ s.t. } \quad F(x)^T(y-x) \geq 0 \quad \forall y \in K(x). \qquad (\text{QVI}(K,F))$$

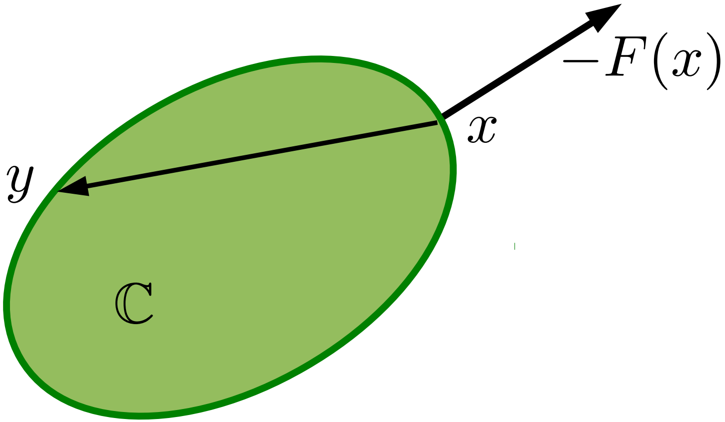
A related **variational inequality** (VI) is the following

$$\text{Find } x \in \mathbb{C} \text{ s.t. } \quad F(x)^T(y-x) \geq 0 \quad \forall y \in \mathbb{C}. \qquad (\text{VI}(\mathbb{C},F))$$

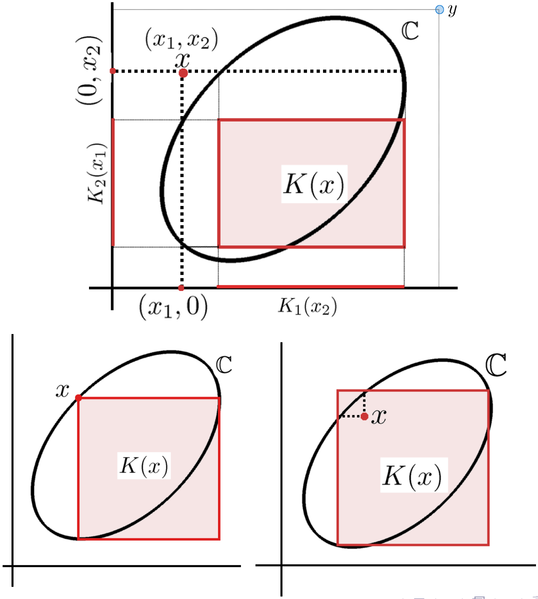
$$x \in \mathcal{R}(x) \iff x \in \text{SOL}(\text{QVI}(K,F)) \qquad x \in \Upsilon(x) \iff x \in \text{SOL}(\text{VI}(\mathbb{C},F)).$$

- $\text{SOL}(\text{VI}(\mathbb{C},F)) \subseteq \text{SOL}(\text{QVI}(K,F))$ was rediscovered later by Facchinei et al. in 2007 [Facchinei et al., 2007]. Solutions of $\text{VI}(\mathbb{C},F)$ were called "variational equilibria"

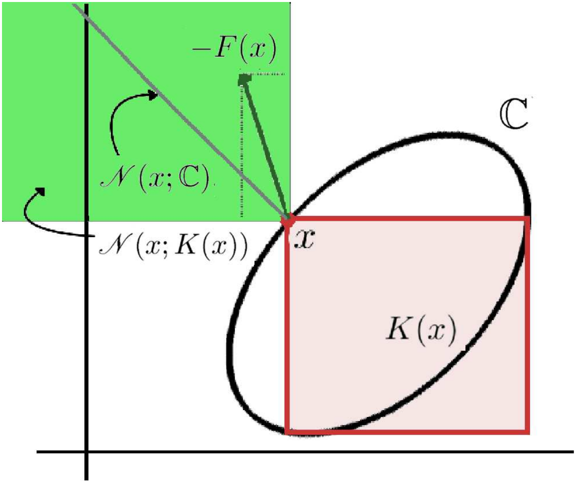
Variational inequality



Structure of K



Quasi-variational inequality



Two kinds of equilibria: KKT conditions

$\mathrm{A}_i(x^{-i})$	$\underset{x_i}{\text{minimize}}$	$\varphi_i(x_i; x^{-i})$	
	subject to	$c(x_i; x^{-i}) \geq 0,$	$: \lambda_i$
		$x_i \geq 0.$	

Shared constraint game with $\mathbb{C} = \{x \mid c(x) \geq 0, x \geq 0\}$

NE	VE(NNE)
$0 \leq x_i \perp \nabla_i \varphi_i(x) - \lambda_i^T \nabla_i c(x) \geq 0$	$0 \leq x_i \perp \nabla_i \varphi_i(x) - \lambda^T \nabla_i c(x) \geq 0$
$0 \leq \lambda_i \perp c(x) \geq 0, \quad \forall\ i \in \mathcal{N}.$	$0 \leq \lambda \perp c(x) \geq 0, \quad \forall\ i \in \mathcal{N}.$

(for $u, v \in \mathbb{R}^n$, $0 \leq u \perp v \geq 0 \equiv u, v \geq 0$ and $u_j v_j = 0,\ j = 1, \cdots, n$)

Noncompact $\mathbb{C}$

- There is a more general principle at play
- Solutions of $\mathrm{VI}(\mathbb{C}, F)$ and $\mathrm{QVI}(K, F)$ are related in a more intimate manner

When $\mathbb{C}$ is not compact, solutions to $\mathrm{QVI}(K, F)$ need not exist. However for a large class of problems, the following is true [Kulkarni and Shanbhag, 2012a]:

- If *any* Nash equilibrium exists, then a solution to $\mathrm{VI}(\mathbb{C}, F)$ also exists. i.e.,

$$\mathrm{SOL}(\mathrm{QVI}(K, F)) \neq \varnothing \iff \mathrm{SOL}(\mathrm{VI}(\mathbb{C}, F)) \neq \varnothing.$$

- In such a situation solutions of $\mathrm{VI}(\mathbb{C}, F)$ are called a refinement of the Nash equilibrium

Definition (Refinement)

A refinement of the set of equilibria of a game is a subset satisfying a certain rule where this rule has the property that any game with a nonempty set of equilibria also possesses an equilibrium satisfying this rule.

- Refined equilibria have some additional properties that make them more attractive. Provide a way of selecting one or few of the many equilibria a game may have. Refinements of **Nash equilibria in matrix games**: trembling hand perfect equilibria [Selten, 1975], proper equilibria [Myerson, 1978] etc (see [Başar and Olsder, 1999]).

More about the refinement

- Such games often have a manifold of NE – can we select a subclass from these with additional properties?
- The Lagrange multipliers can be interpreted as the "price" charged on a player by an administrator.
- The equilibrium with non-shared multipliers is an equilibrium resulting from "discriminatory prices"
- But often the situation modeled makes it unrealistic for the administrator to be able to distinguish between various users
- The VE is really the "**right**" equilibrium.
- But ··· does a VE **always exist**? Can an administrator charge a uniform price across all users to enforce equilibrium?
- If the VE is a refinement of the GNE, then an equilibrium with discriminatory prices exists, **if and only if** one with uniform prices also exists.

Main results

- $\widetilde{\mathbf{F}}_K^{\text{nat}}$ = *natural map* of $\text{QVI}(K, F)$.

$$\widetilde{\mathbf{F}}_K^{\text{nat}}(x) = 0 \iff x \in \text{SOL}(\text{QVI}(K, F)) \iff x \text{ is a GNE}$$

- $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$ = *natural map* of $\text{VI}(\mathbb{C}, F)$.

$$\mathbf{F}_{\mathbb{C}}^{\text{nat}}(x) = 0 \iff x \in \text{SOL}(\text{VI}(\mathbb{C}, F)) \iff x \text{ is a VE}$$

- $\deg(f, \Omega, p)$: Brouwer degree of f w.r.t. p over Ω
- *well defined* if $p \notin f(\partial\Omega)$
- $\deg(f, \Omega, p) \neq 0 \implies \exists\, x \in \Omega$ s.t. $f(x) = p$ (**note:** converse is false)

Theorem

Let Ω *be an open bounded set such that* $\overline{\Omega} \subseteq \text{dom}(K)$. *If* $0 \notin \widetilde{\mathbf{F}}_K^{\text{nat}}(\partial\Omega)$, *then there is a homotopy* $H(t, x)$ *such that* $0 \notin H([0,1], \partial\Omega)$ *and* $H(1, \cdot) = \widetilde{\mathbf{F}}_K^{\text{nat}}$ *and* $H(0, \cdot) = \mathbf{F}_{\mathbb{C}}^{\text{nat}}$. *Furthermore,*

$$\deg(\widetilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0) = \deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0).$$

Thus if $\text{SOL}(\text{QVI}(K, F)) \neq \varnothing \implies \deg(\widetilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0) \neq 0$ *then* $\text{SOL}(\text{QVI}(K, F)) \neq \varnothing \implies \text{SOL}(\text{VI}(\mathbb{C}, F)) \neq \varnothing$.

Main results

- The result says that $\widetilde{\mathbf{F}}_K^{\text{nat}}$ and $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$ can be transformed smoothly without losing certain properties
- They are equivalent upto their Brouwer degree
- These conditions are also necessary if one assumes F to be monotone
- There are analogous and more powerful results in the 'primal-dual' $x-\lambda$ space
- More in [Kulkarni and Shanbhag, 2012a], [Kulkarni and Shanbhag, 2009], [Kulkarni and Shanbhag, 2012b].

Beyond convexity

- Rosen's argument, i.e., $\text{Fix}(\Upsilon) \subseteq \text{Fix}(\mathcal{R})$ works even when $\mathbb{C}$ is not convex
- **First order equilibria:** If $\mathbb{C}$ is not convex but given via algebraic constraints, the above results apply for "first order equilibria" or Nash stationary points – i.e., those points at which KKT conditions for the game hold.
- **More general fixed point theorems:**

Eilenberg-Montgomery FPT

If X is a compact acyclic absolute neighbourhood retract and $T : X \to 2^X$ takes acyclic values, then $\text{Fix}(\text{T}) \neq \varnothing$.

- Theory of retracts [Borsuk, 1967], [Hu, 1965]
- Example: X is contractible and and T is contractible-valued.
- **Another argument without fixed point theory:** more on this later

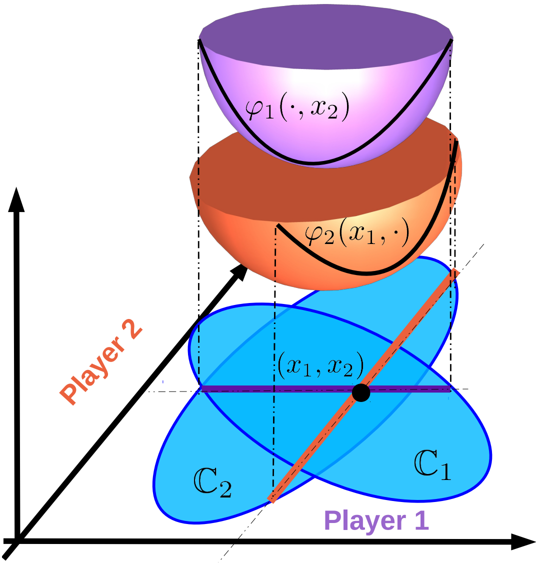
Beyond shared constraints

Suppose there exist $\mathbb{C}_i, i \in \mathcal{N}$ such that player i solves

$$\begin{array}{ll} \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{*,-i}) \\ \text{subject to} & (x_i, x^{*,-i}) \in \mathbb{C}_i. \end{array}$$

- Arrow-Debreu [Arrow and Debreu, 1954] *abstract economy.*
- Today also called *generalized Nash game*
- Rosen's argument does not work
- It seems continuity of $K_i(x^{-i}) = \{x_i | (x_i, x^{-i}) \in \mathbb{C}_i\}$ is required
- However, there are interesting work-arounds. More later...

Generalized Nash game



Multi-leader multi-follower games

Setting

- Set of players categorized as "leaders" and "followers"
- Followers take decisions with the knowledge of the decisions of the leaders
- Amongst themselves, followers play a noncooperative game*
- Leaders choose their decisions while anticapting the response of the followers to these decisions
- Amongst themselves, leaders play a noncooperative game

Applications

- Power markets with sequential clearings
  1. Firms broadcast their decisions
  2. Spot market clears taking the firms' decisions for granted.
  0. Firms decide what decisions to broadcast based on the Nash equilibrium in the spot market and are themselves in Nash equilibrium
- Multiple competing servers; followers decide which service to choose.

* *not really needed; follower behavior could be obtained from any other logic, so long as it is "common" to all leaders.*

Leader's problem

- $\mathcal{N} = \{1, \cdots, N\}$ = set of leaders, objectives φ_i and strategies x_i

$$x = (x_1, \cdots, x_N), \qquad x^{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N)$$

- y_i = follower equilibrium conjectured by leader i

$$y = (y_1, \cdots, y_N), \qquad y^{-i} = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_N)$$

- $\mathcal{S}(x)$ = set of follower equilibria when leaders play x. So $y_i \in \mathcal{S}(x)$ for all $i \in \mathcal{N}$.

Leader-follower game

- Technically, the action of the i^{th} leader is only x_i. But the choice of x_i depends on y_i; consequently y_i also must be interpreted as an action
- Optimistic formulation

$$\mathrm{L}_i(x^{-i}) \qquad \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \quad \text{subject to} \quad x_i \in X_i, \; y_i \in \mathcal{S}(x).$$

- **Equilibrium:** (x, y) such that $(x_i, y_i) \in \mathrm{SOL}(L_i(x^{-i}))$ for all $i \in \mathcal{N}$.

Challenges

Existence

- Even simple games do not admit equilibria. E.g., [Pang and Fukushima, 2005]: 2 leaders, 1 follower. $X_1 = X_2 = [0, 1]$

$$\textbf{Leaders: } \varphi_1(x_1, y_1) = \tfrac{1}{2}x_1 + y_1, \quad \varphi_2(x_2, y_2) = -\tfrac{1}{2}x_2 - y_2$$

$$\textbf{Follower: } \bar{y} = \arg\min_{\bar{y}\geq 0}\{\bar{y}(-1 + x_1 + x_2) + \tfrac{1}{2}\bar{y}^2\} = \max\{0, 1 - x_1 - x_2\}$$

Computation

- Inordinately hard to compute the equilibrium. No convergent schemes.
  ... although the multi-leader-follower problem is a sensible mathematical model with a well-defined solution concept, its high level of complexity and technical hardship make it a computationally intractable problem. [Pang and Fukushima, 2005]

Meaningfulness/Usefulness

- Later ...

New approach

Standard approach

$$\mathrm{L}_i(x^{-i}) \qquad \begin{array}{ll} \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\ \text{subject to} & \begin{array}{l} x_i \in X_i, \\ y_i = \mathcal{S}(x). \end{array} \end{array}$$

Explicit substitution of y_i leading to tedious calculations.

New approach

- **Conceptual issue** [Kulkarni and Shanbhag, IEEE CDC 2013]
- **Mathematical structure** [Kulkarni and Shanbhag, Set Valued and Variational Analysis, 2014]
- Clean result on **existence of equilibria** [Kulkarni and Shanbhag, IEEE TAC 2014]
- **New approach to general dynamic games** [Abraham and Kulkarni, under review with IEEE TAC]

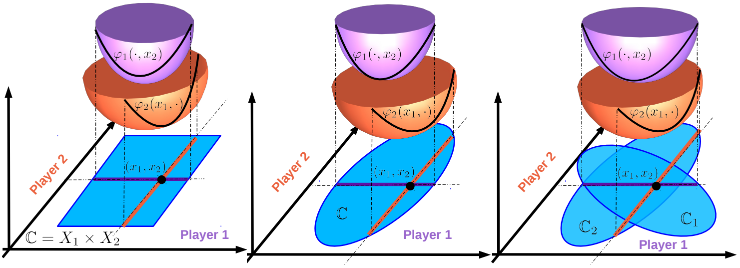
Properties

Nonconvexity of constraints

$$K_i(x^{-i}) = \{(x_i, y_i) \mid y_i \in \mathcal{S}(x_i; x^{-i})\}$$

is typically nonconvex for each x^{-i}.

Coupled constraint game



Reaction map not upper semicontinuous

$$\mathcal{R}_i(x^{-i}) = \mathrm{SOL}(L_i(x^{-i})) = \arg \min_{y_i \in \mathcal{S}(x_i, x^{-i})} \varphi_i(x_i, y_i; x^{-i}).$$

Meaningfulness?

[Kulkarni and Shanbhag, 2013]

Meaningfulness/usefulness

- When $\mathcal{S}(\cdot)$ is multivalued, at equilibrium leaders may disagree on their conjectures about the follower equilbirum. i.e., $y_i \neq y_j$ for some i, j.
- If the y_i, y_j represent a physical value, one is lead to ask if such an equilibrium is even meaningful/useful (e.g., electricity markets)

- We can attempt to resolve this as follows.

Ex-post consistency

- Suppose we ask for an equilibrium such that $y_i = y_j$ for all $i, j \in \mathcal{N}$i.e.,

$$(x_i, y_i) \in \mathrm{SOL}(L_i(x^{-i})) \ \forall i \in \mathcal{N} \qquad \text{and} \qquad y_i = y_j \ \forall i, j \in \mathcal{N}.$$

Problem with this...

- Over determined system
- Equilibria in the standard sense rarely exist
- Too strong...

An alternative: ex-ante consistency

Consistency of conjectures

- Impose consistency as part of the decision problem of each leader
- New game:

$$\mathrm{L}_i^{cc}(x^{-i}, y^{-i}) \qquad \begin{aligned} &\underset{x_i, y_i}{\text{minimize}} && \varphi_i(x_i, y_i; x^{-i}) \\ &\text{subject to} && x_i \in X_i, \\ & && y_i \in \mathcal{S}(x), \\ & && y_i = y_j, \ \forall i, j. \end{aligned}$$

Consequences

An alternative: ex-ante consistency

Consistency of conjectures

- Impose consistency as part of the decision problem of each leader
- New game:

$$\mathrm{L}_i^{cc}(x^{-i}, y^{-i}) \qquad \begin{array}{ll} \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\ \text{subject to} & \begin{array}{l} x_i \in X_i, \\ y_i \in \mathcal{S}(x), \\ y_i = y_j, \ \forall i, j. \end{array} \end{array}$$

Consequences

- *Consistency:* of conjectures at equilibrium holds trivially (even when $\mathcal{S}(\cdot)$ is multivalued)

An alternative: ex-ante consistency

Consistency of conjectures

- Impose consistency as part of the decision problem of each leader
- New game:

$$\mathrm{L}_i^{cc}(x^{-i}, y^{-i}) \qquad \begin{array}{ll} \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\ \text{subject to} & x_i \in X_i, \\ & y_i \in \mathcal{S}(x), \\ & y_i = y_j, \ \forall i, j. \end{array}$$

Consequences

- *Consistency:* of conjectures at equilibrium holds trivially (even when $\mathcal{S}(\cdot)$ is multivalued)
- *Retaining original equilibria:* If $\mathcal{S}(\cdot)$ is single-valued, equilibria of original game are equilibria of the new game

An alternative: ex-ante consistency

Consistency of conjectures

- Impose consistency as part of the decision problem of each leader
- New game:

$$\mathrm{L}_i^{cc}(x^{-i}, y^{-i}) \qquad \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \qquad \text{subject to} \quad x_i \in X_i, \quad y_i \in \mathcal{S}(x), \quad y_i = y_j, \ \forall i, j.$$

Consequences

- *Consistency:* of conjectures at equilibrium holds trivially (even when $\mathcal{S}(\cdot)$ is multivalued)
- *Retaining original equilibria:* If $\mathcal{S}(\cdot)$ is single-valued, equilibria of original game are equilibria of the new game
- *Existence:* Equilibria exist under milder conditions; in particular new version of Pang and Fukushima example admits an equilibrium.

An alternative: ex-ante consistency

Consistency of conjectures

- Impose consistency as part of the decision problem of each leader
- New game:

$$\mathrm{L}_i^{cc}(x^{-i}, y^{-i}) \qquad \begin{array}{ll} \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\ \text{subject to} & x_i \in X_i, \\ & y_i \in \mathcal{S}(x), \\ & y_i = y_j, \ \forall i, j. \end{array}$$

Consequences

- *Consistency:* of conjectures at equilibrium holds trivially (even when $\mathcal{S}(\cdot)$ is multivalued)
- *Retaining original equilibria:* If $\mathcal{S}(\cdot)$ is single-valued, equilibria of original game are equilibria of the new game
- *Existence:* Equilibria exist under milder conditions; in particular new version of Pang and Fukushima example admits an equilibrium.
- *Computation:* Much easier to compute; natural schemes converge

Existence

Theorem

- *If $\mathcal{S}(\cdot)$ is single-valued, every equilibrium of the original game is an equilibrium of the new game.*
- *The constraints of the new game,*

$$\Omega_i(x^{-i}, y^{-i}) = \{(x_i, y_i) \mid y_i \in \mathcal{S}(x), y_i = y_j \forall j \in \mathcal{N}\}$$

form a shared constraint. i.e., $\exists$ a set $\mathcal{F}$ such that for all $i \in \mathcal{N}$,

$$(u_i, v_i) \in \Omega_i(x^{-i}, y^{-i}) \iff (u_i, x^{-i}, v_i, y^{-i}) \in \mathcal{F}.$$

$$\mathcal{F} = \{(x, y) \mid x_i \in X_i, y_i \in \mathcal{S}(x) \forall i \in \mathcal{N}, y_i = y_j \forall i, j \in \mathcal{N}\}.$$

The constraints of the original game do not have this structure.

Theorem

Suppose the objectives of the leaders $\{\varphi_i\}_{i \in \mathcal{N}}$ admit a potential function π. Then any minimizer of

$$\min_{(x,y) \in \mathcal{F}} \pi(x, y)$$

is an equilibrium of the new game. Thus if π is continuous and $\mathcal{F}$ is compact, the new game admits an equilibrium.

The Pang and Fukushima example, revisited

$$\text{L}_1^{cc}(x_2, y_2) \quad \underset{x_1, y_1}{\text{minimize}} \quad \varphi_1(x_1, y_1) = \tfrac{1}{2}x_1 + y_1$$
$$\text{subject to} \quad x_1 \in [0,1], y_1 = \max\{0, 1 - x_1 - x_2\}, \quad y_1 = y_2.$$

$$\text{L}_2^{cc}(x_1, y_1) \quad \underset{x_2, y_2}{\text{minimize}} \quad \varphi_2(x_2, y_2) = -\tfrac{1}{2}x_2 - y_2$$
$$\text{subject to} \quad x_2 \in [0,1], y_2 = \max\{0, 1 - x_1 - x_2\}, \quad y_1 = y_2.$$

Potential game with $\pi = \varphi_1 + \varphi_2$

$$\mathcal{F} = \left\{(x,y) | x \in [0,1]^2, y_1 = y_2 = \max(0, 1 - x_1 - x_2)\right\}$$

$$\arg \min_{(x,y)\in\mathcal{F}} \pi = \arg \min_{(x,y)\in\mathcal{F}} \tfrac{1}{2}x_1 + y_1 - \tfrac{1}{2}x_2 - y_2 = ((0,1),(0,0)),$$

Easy to check that $((0,1),(0,0))$ is an equilibrium.

What happened?

Ex-post v/s ex-ante consistency of conjectures

- With ex-ante consistency, conjectures are consistent not just at equilibrium
- Equivalently, stability is sought only against those deviations in conjectures that themselves consistent

Food for thought...

- Not only have we solved meaningfulness, we have also to some extent solved existence
- Consistency provides meaningfulness, but prima facie there is no reason to think it will also facilitate existence of equilibria
- Does consistency of conjectures rid the problem of some inherent pathology?

Shared constraints

[Kulkarni and Shanbhag, 2014b]

- **Leaders sharing all equilibrium constraints $\mathscr{E}^{ae}$**
- Require that all conjectures about follower equilibria be seen by all players

$$\mathrm{L}_i^{ae}(x^{-i}, y^{-i}) \qquad \begin{array}{ll} \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\ \text{subject to} & x_i \in X_i, \\ & y_j \in \mathcal{S}(x) \qquad j = 1, \cdots, N. \end{array}$$

- Shared constraint game with constraint $\mathcal{F} = \{(x, y) \mid x \in X, y_i \in \mathcal{S}(x)\ i = 1, \cdots, N\}$.

Theorem

- *Every equilibrium of the conventional formulation is an equilibrium of $\mathscr{E}^{ae}$.*
- *If the game is a potential game, every minimizer of the potential function is an equilibrium of $\mathscr{E}^{ae}$.*

Existence of equilibria

[Kulkarni and Shanbhag, 2014a]

Definition (Quasi-potential game)

(i) For $i = 1, \cdots, N$, there exist functions $\phi_1(x), \cdots, \phi_N(x)$ and a function $h(x, y_i)$ such that each player i's objective $\varphi_i(\cdot)$ is given as $\varphi_i(x_i, y_i; x^{-i}) \equiv \phi_i(x) + h(x, y_i)$.

(ii) There exists a function $\pi(\cdot)$ such that for all $i = 1, \cdots, N$, and for all $x \in X$ and $x_i' \in X_i$, we have $\phi_i(x_i; x^{-i}) - \phi_i(x_i'; x^{-i}) = \pi(x_i; x^{-i}) - \pi(x_i'; x^{-i})$.

Theorem

Consider a quasi-potential multi-leader multi-follower game. If (x, w) is a global minimizer of $\mathrm{P}^{\text{quasi}}$, then (x, y), where $y_i = w$ for all $i \in \mathcal{N}$, is a global equilibrium of the game.

$$\mathrm{P}^{\text{quasi}} \qquad \begin{array}{ll} \underset{x,w}{\text{minimize}} & \pi(x) + h(x, w) \\ \text{subject to} & (x, w) \in \mathcal{F}^{\text{quasi}} \end{array}$$

$$\mathcal{F}^{\text{quasi}} \triangleq \{ (x, w) \in \mathbb{R}^n \mid x_i \in X_i, \ i = 1, \cdots, N, w \in \mathcal{S}(x) \}.$$

Indeed, there exists an equilibrium with ***consistent conjectures***.

The Pang and Fukushima example, revisited

$$\mathrm{L}_1(x_2)\ \underset{x_1,y_1}{\text{minimize}} \quad \varphi_1(x_1,y_1) = \tfrac{1}{2}x_1 + y_1 \quad \text{subject to} \quad x_1 \in [0,1],\ y_1 = \max\{0, 1 - x_1 - x_2\}$$

$$\mathrm{L}_2(x_1)\ \underset{x_2,y_2}{\text{minimize}} \quad \varphi_2(x_2,y_2) = -\tfrac{1}{2}x_2 - y_2 \quad \text{subject to} \quad x_2 \in [0,1],\ y_2 = \max\{0, 1 - x_1 - x_2$$

Quasi-otential game with $\pi(x_1,x_2) = \frac{1}{2}(x_1 + x_2)$ and $h(x,w) = -w$

$$\mathcal{F}^{\text{quasi}} = \left\{(x,w) | x \in [0,1]^2, w = \max(0, 1 - x_1 - x_2)\right\}$$

$$\arg \min_{(x,w)\in\mathcal{F}^{\text{quasi}}} \pi(x) + h(x;w) = (0,0,1),$$

Easy to check that $(x_1,x_2,y_1,y_2) = (0,0,1,1)$ is an equilibrium.

The Pang and Fukushima example, revisited

$$\mathrm{L}_1(x_2)\ \underset{x_1,y_1}{\text{minimize}} \quad \varphi_1(x_1,y_1) = \tfrac{1}{2}x_1 - y_1 \quad \text{subject to} \quad x_1 \in [0,1],\ y_1 = \max\{0, 1 - x_1 - x_2\}$$

$$\mathrm{L}_2(x_1)\ \underset{x_2,y_2}{\text{minimize}} \quad \varphi_2(x_2,y_2) = -\tfrac{1}{2}x_2 - y_2 \quad \text{subject to} \quad x_2 \in [0,1],\ y_2 = \max\{0, 1 - x_1 - x_2$$

Quasi-otential game with $\pi(x_1,x_2) = \frac{1}{2}(x_1 + x_2)$ and $h(x,w) = -w$

$$\mathcal{F}^{\text{quasi}} = \left\{(x,w) | x \in [0,1]^2, w = \max(0, 1 - x_1 - x_2)\right\}$$

$$\arg \min_{(x,w)\in\mathcal{F}^{\text{quasi}}} \pi(x) + h(x;w) = (0,0,1),$$

Easy to check that $(x_1,x_2,y_1,y_2) = (0,0,1,1)$ is an equilibrium.

Discrete Time Dynamic Game (open loop information structure)

- Player Set, $\mathcal{N} = \{1, 2, ..., N\}$
- Stage Set, $\mathcal{K} = \{1, 2, ..., K\}$
- State Space, X_k , $k \in \mathcal{K}$
- Control Space, U_k^i, $i \in \mathcal{N}$, $k \in \mathcal{K} \backslash \{K\}$. $U_k^{-i} \triangleq \prod_{j \in \mathcal{N}, j \neq i} U_k^j$ - Control space of adversaries
- State Equation

$$x_{k+1} = f_k(x_k, u_k^1, u_k^2, ..., u_k^N)$$

- Cost Functional of player $i \in \mathcal{N}$ $J^i : (U_1^1 \times ... \times U_1^N) \times (X_2^i \times X_2^{-i} \times U_2^1 \times ... \times U_2^N) \times ... \times (X_{K-1}^i \times X_{K-1}^{-i} \times U_{K-1}^1 \times ... \times U_{K-1}^N) \times (X_K^i \times X_K^{-i}) \longrightarrow \mathbb{R}$

$$\begin{array}{lll} \mathrm{P}_i(u^{-i}) & \underset{u^i}{\text{minimize}} & J^i(u^i, u^{-i}, x) \\ & \text{subject to} & x_{k+1} = f_k(x_k, u_k^i, u_k^{-i}) \quad \forall\ k \in \mathcal{K}, \\ & & u_k^i \in U_k^i \quad \forall\ k \in \mathcal{K}. \end{array}$$

- **Standard approach:** substitute state equation into cost. Tractable only for LQ games.
- **New approach:** Leave state equation as a constraint and consider x_k as a *decision variable*

State Conjecture formulation

[Abraham and Kulkarni, 2015]

Let x_k^i denote player i's conjecture about the state.

$$\mathrm{P}_i(u^{-i}, x^{-i}) \qquad \underset{u^i, x^i}{\text{minimize}} \quad J^i(u^i, x^i; u^{-i}, x^{-i})$$
$$\text{subject to} \quad \begin{aligned} x_{k+1}^i &= f_k(x_k^i, u_k^i, u_k^{-i}) \quad \forall\ k \in \mathcal{K}, \\ u_k^i &\in U_k^i \quad \forall\ k \in \mathcal{K}, \end{aligned}$$

Theorem

- *If the game has a quasi-potential structure, then any minimizer of the quasi-potential function over a suitably defined set is an equilibrium*
- *Certain classes of LQ games admit quasi-potential functions*

Implications

- Clean existence result, generalizes the theory beyond LQ games

Consistency of state conjectures

$$
\begin{array}{lll}
\mathrm{P}_i(u^{-i}, x^{-i}) & \underset{u^i, x^i}{\text{minimize}} & J^i(u^i, x^i; u^{-i}, x^{-i}) \\
 & & x^i_{k+1} = f_k(x^i_k, u^i_k, u^{-i}_k) \quad \forall\ k \in \mathcal{K}, \\
 & \text{subject to} & u^i_k \in U^i_k \quad \forall\ k \in \mathcal{K}, \\
 & & x^i_k = x^j_k \quad \forall j \in \mathcal{N}, k \in \mathcal{K}.
\end{array}
$$

Theorem

- *The above game is a shared constraint game*
- *If the game admits a potential function and spaces X, U are compact and functions f_k are continuous, the game admits an equilibrium*
- *Any equilibrium of the original game is an equilibrium of the new game*

- More can be said – e.g., ϵ-equilibrium. See more in [Abraham and Kulkarni, 2015]

Abraham, M. P. and Kulkarni, A. A. (2015).
New results on existence of open loop Nash equilibria in discrete time dynamic games.
to be submitted to IEEE Transactions on Automatic Control.

Arrow, K. and Debreu, G. (1954).
Existence of an equilibrium for a competitive economy.
Econometrica, 22(3):265–290.

Başar, T. and Olsder, G. (1999).
Dynamic Noncooperative Game Theory.
SIAM, Philadelphia, second edition.

Borsuk, K. (1967).
Theory of retracts.
Polish Scientific Publishers, Warsaw, 1st edition.

Facchinei, F., Fischer, A., and Piccialli, V. (2007).
On generalized Nash games and variational inequalities.
Operations Research Letters, 35(2):159–164.

Hu, S.-T. (1965).
Theory of retracts.
Wayne State University Press, Detroit, first edition.

Kulkarni, A. A. and Shanbhag, U. V. (2009).
New insights on generalized nash games with shared constraints: Constrained and variational equilibria.
In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 151–156.

Kulkarni, A. A. and Shanbhag, U. V. (2012a).
On the variational equilibrium as a refinement of the generalized Nash equilibrium.
Automatica, 48(1):45–55.

Kulkarni, A. A. and Shanbhag, U. V. (2012b).
Revisiting generalized Nash games and variational inequalities.
Journal of Optimization Theory and Applications, 154(1):1–12.

Kulkarni, A. A. and Shanbhag, U. V. (2013).
On the consistency of leaders' conjectures in hierarchical games.
In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pages 1180–1185.

Kulkarni, A. A. and Shanbhag, U. V. (2014a).
An existence result for hierarchical Stackelberg v/s Stackelberg games.

accepted subject to minor revisions by the IEEE Transactions on Automatic Control.

Kulkarni, A. A. and Shanbhag, U. V. (2014b).
A shared-constraint approach to multi-leader multi-follower games.
Set-Valued and Variational Analysis, 22(4):691–720.

Myerson, R. B. (1978).
Refinements of the Nash equilibrium concept.
International Journal of Game Theory, 7(2):73–80.

Pang, J.-S. and Fukushima, M. (2005).
Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games.
Computational Management Science, 2(1):21–56.

Rosen, J. B. (1965).
Existence and uniqueness of equilibrium points for concave N-person games.
Econometrica, 33(3):520–534.

Selten, R. (1975).
Reexamination of the perfectness concept for equilibrium points in extensive games.
International Journal of Game Theory, 4(1):25–55.