Markets with Production and constant number of goods

Jugal Garg and Ravi Kannan [Nikhil Devanur]

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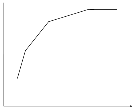

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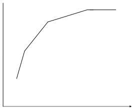

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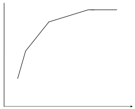

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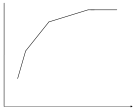

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- 2 models : (i) Each agent comes with a fixed amount of money or (ii) a fixed bundle of goods.

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- Note : If prices are given, each agent's problem is an LP – maximize min of several linear function. If prices are unknowns, problem is non-linear – (price)(amount of good bought).

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  - Leontief utilities (special case of PLC)

$$U_i(x_i) = min_j x_{ij}/\phi_{ij}.$$

Our Results

- Devanur, K. Polynomial time algorithm for finding an exact equilibrium when the number of goods in constant. We also do the same for the case when the number of agents is constant provided the utilities are separable –

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- Jugal Garg, K.
  - Constant number of goods now with Production - poly time
  - Reduction from market with Production, PLC utilities to one with no production and PLC utilities with same set of equilibria (1-1 correspondance)

Overview of method

- Step 1: Cell decomposition: Divide price space $\mathbf{R}_+^m$ into small cells, either by hyperplanes or polynomial surfaces so that (intuitively), the "order" of the pieces in the PLC utilities is the same for all price vectors in the cell.

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- Step II : In each cell, either find a price that has a market clearing allocation, or certify that no such price exists. For this, one solves a system of linear/polynomial equations involving a constant number of variables.

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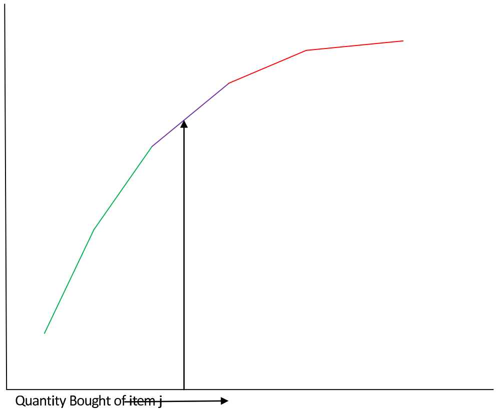
- Quick Proof: If the agent was buying something with lower MUPUC than something the agent was not buying, then transfer some money from the former to the latter to gain. [Separable.]


Quantity Bought of item j

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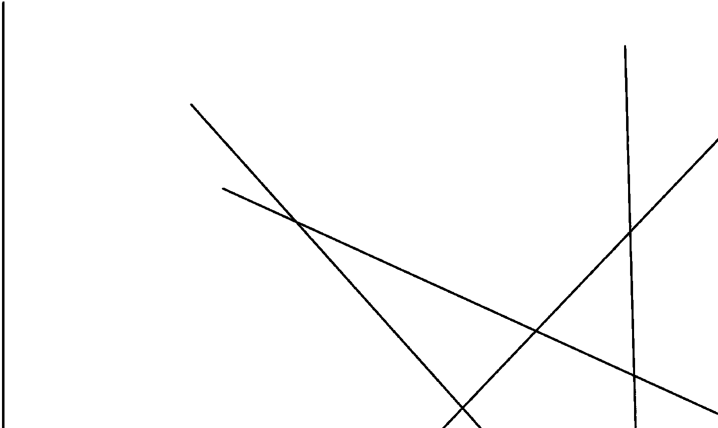
- N hyperplanes in m space divide space into at most $\binom{N}{m}$ non-empty cells !! They can be found.



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$$\text{Clearing Good } j: \sum_i f_{ij} = p_j(\text{residual amount of good } j)$$

$$\text{Agent } i: \sum_j f_{ij} = \text{ residual budget of } i$$

$$0 \leq f_{ij}$$

Linear Program !!

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- We can find some M linear constraints on p so that for all p satisfying those constraints, there is a single face F so that the set of optimal solutions is $F \cap$ budget. (LP Duality).
  $M \leq$POLY for $m \in O(1)$.

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- N degree d polynomial inequalities in $2m$ space produce at most N^{md} cells.

Production

- Now include a number of factories. Each factory has a *production set* of pairs (x, y), where x, y are each a bundles (of goods). With x as raw materials, factory can produce y. The production set is convex. Each factory maximizes its profit independent of others.

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- Equilibrium: Prices at which individually optimized x_i, y_l, z_l clear the market.

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- Proceed as earlier.... (Some more technical issues...)