Robust Multi-Product Pricing Optimization with Experiments

Chung-Piaw Teo

National University of Singapore

Cong Cheng, Northeastern University, China
Karthik Natarajan, Singapore Univ. of Tech. and Design
Zhenzhen Yan, National University of Singapore

Problem: Can we optimize the prices to maximize profit for the seller?

The utility that the customer gets from purchasing a product is modeled as:

$$\tilde{U}_j = W_j - p_j + \tilde{\epsilon}_j \quad j \in \mathcal{N} \cup \{0\}, \tag{1}$$

where p_j is the price of the product and W_j is the observable utility associated with other attributes of the product j. The random error term $\tilde{\epsilon}_j$ models the unobservable characteristics of the utility function.

Problem: Can we optimize the prices to maximize profit for the seller?

The utility that the customer gets from purchasing a product is modeled as:

$$\tilde{U}_j = W_j - p_j + \tilde{\epsilon}_j \quad j \in \mathcal{N} \cup \{0\}, \tag{1}$$

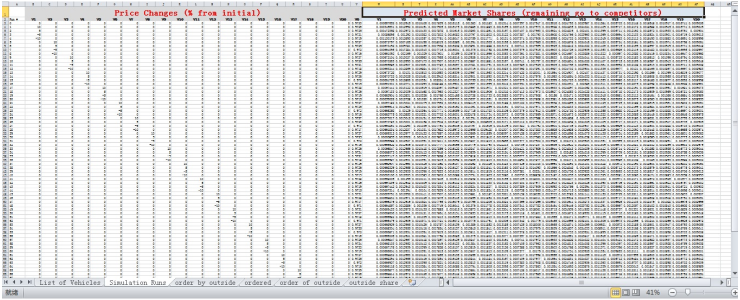
where p_j is the price of the product and W_j is the observable utility associated with other attributes of the product j. The random error term $\tilde{\epsilon}_j$ models the unobservable characteristics of the utility function.

For a given price vector $p = (p_1, \ldots, p_n)$, the probability that a customer selects product j is:

$$P_j(p) = P\left(W_j - p_j + \tilde{\epsilon}_j \geq \max_{k \in \mathcal{N} \cup \{0\}} (W_k - p_k + \tilde{\epsilon}_k)\right).$$

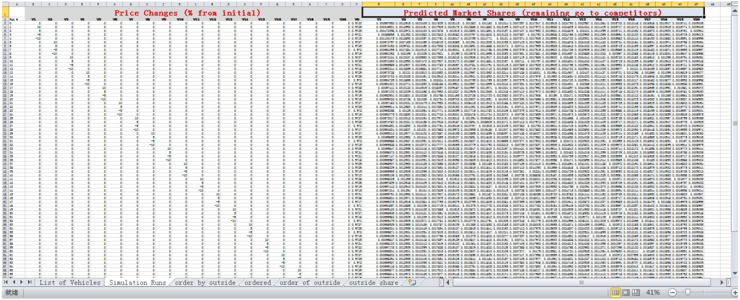
Challenge:

Companies build elaborate market share simulation model to evaluate the perofrmance of pricing proposals...



Challenge:

Companies build elaborate market share simulation model to evaluate the perofrmance of pricing proposals...



Can we learn from these experiments to obtain the optimal prices?

Challenge:

Companies build elaborate market share simulation model to evaluate the perofrmance of pricing proposals...

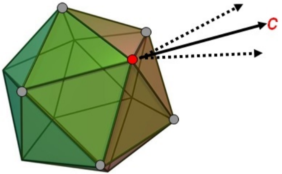
	A	B	C	D	E	F	G	H	I	J	K
1							Price Changes				
2	Run #	V1	V2	V3	V4	V5	V6	V7	V8	V9	V1
3	0	0	0	0	0	0	0	0	0	0	
4	1	10	0	0	0	0	0	0	0	0	
5	2	5	0	0	0	0	0	0	0	0	
6	3	-5	0	0	0	0	0	0	0	0	
7	4	-10	0	0	0	0	0	0	0	0	
8	5	0	10	0	0	0	0	0	0	0	
9	6	0	5	0	0	0	0	0	0	0	
10	7	0	-5	0	0	0	0	0	0	0	
11	8	0	-10	0	0	0	0	0	0	0	
12	9	0	0	10	0	0	0	0	0	0	
13	10	0	0	5	0	0	0	0	0	0	
14	11	0	0	-5	0	0	0	0	0	0	
15	12	0	0	-10	0	0	0	0	0	0	
16	13	0	0	0	10	0	0	0	0	0	
17	14	0	0	0	5	0	0	0	0	0	
18	15	0	0	0	-5	0	0	0	0	0	
19	16	0	0	0	-10	0	0	0	0	0	
20	17	0	0	0	0	10	0	0	0	0	
21	18	0	0	0	0	5	0	0	0	0	
22	19	0	0	0	0	-5	0	0	0	0	
23	20	0	0	0	0	-10	0	0	0	0	

Theme: Predicting Choices in Optimization and Games

Solve general random mixed 0-1 LP problem under objective uncertainty:

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x} \in \mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j,$$

$$\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T \mathbf{x} = b_i, \forall i, \; x_j \in \{0,1\}, \forall j \in \mathcal{B} \subseteq \{1,\ldots,n\}, \; \mathbf{x} \geq 0$$



(eg. $\tilde{\mathbf{c}} \sim N(\mu, \mathbf{\Sigma})$)

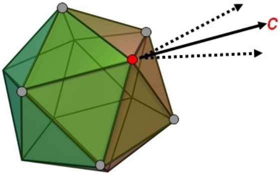
Goal: Design "$\tilde{c}$" to obtain desired x.

Theme: Predicting Choices in Optimization and Games

Solve general random mixed 0-1 LP problem under objective uncertainty:

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x}\in\mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j,$$

$$\mathcal{P} := \{\mathbf{x}\in\mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} = b_i, \forall i, \; x_j \in \{0,1\}, \forall j \in \mathcal{B} \subseteq \{1,\ldots,n\}, \; \mathbf{x} \geq \mathbf{0}$$



(eg. $\tilde{\mathbf{c}} \sim N(\mu, \mathbf{\Sigma})$)

Goal: Design "$\tilde{c}$" to obtain desired x.

What is $P(\mathbf{x}_i(\tilde{\mathbf{c}}) = 1)$?

Example: Discrete Choice

Which product will she buy?

Example: Discrete Choice

Which product will she buy?

Assumes consumer maximizes utility

Example: Discrete Choice

Which product will she buy?

Assumes consumer maximizes utility

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x}\in\mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j, \qquad \tilde{c}_j \text{ utility of product } j$$

$$\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_j \in \{0,1\},\ \forall j\ \mathbf{x} \geq \mathbf{0}\}.$$

Example: Discrete Choice

Which product will she buy?

Assumes consumer maximizes utility

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x} \in \mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j, \qquad \tilde{c}_j \text{ utility of product } j$$

$$\mathcal{P} := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_j \in \{0,1\}, \forall j\ \mathbf{x} \geq \mathbf{0} \right\}.$$

How to model $\tilde{\mathbf{c}}$*?*

Example: Discrete Choice

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x} \in \mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j, \qquad \tilde{c}_j \text{ utility of product } j$$

$$\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_j \in \{0,1\}, \forall j\ \mathbf{x} \geq \mathbf{0}\}.$$

Example: Discrete Choice

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x} \in \mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j, \quad \tilde{c}_j \text{ utility of product } j$$

$$\mathcal{P} := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, \ x_j \in \{0,1\}, \forall j \ \mathbf{x} \geq \mathbf{0} \right\}.$$

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Example: Discrete Choice

$$Z(\tilde{\mathbf{c}}) := \max_{\mathbf{x}\in\mathcal{P}} \sum_{j=1}^{n} \tilde{c}_j x_j, \qquad \tilde{c}_j \text{ utility of product } j$$

$$\mathcal{P} := \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_j \in \{0,1\}, \forall j\ \mathbf{x} \geq \mathbf{0}\right\}.$$

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)
- $P\left(\mathbf{x}_j(\tilde{\mathbf{c}}) = 1\right) = \frac{e^{\beta\cdot\mathbf{A_j}}}{\sum_{k=1}^{n} e^{\beta\cdot\mathbf{A_j}}}$,
- β estimated from observed choices

Issue: Discrete Choice

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Issue: Discrete Choice

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Which product will she buy? What is the outside option?

Issue: Discrete Choice

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Which product will she buy? What is the outside option?

Scale - idosyncratic noise for outside option is different

Heterogeneity - each customer is different

Issue: Discrete Choice

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Which product will she buy? What is the outside option?

Issue: Discrete Choice

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Which product will she buy? What is the outside option?

Example: 5 products. Means and Std Dev of Utilities (independent and normally distributed) of Products are

Item	1	2	3	4	5
Mean	6	6.2	6.4	6.6	6.8
Std Dev	2.00000000	2.00000000	2.00000000	2.00000000	1.00000000

Issue: Discrete Choice

Logit Model

- $\tilde{c}_j = \overbrace{\beta}^{\text{weights}} \cdot \overbrace{\mathbf{A_j}}^{\text{attributes}} + \overbrace{\tilde{\epsilon}_j}^{\text{noise}}$ (eg. i.i.d. Gumbel Distribution)

Which product will she buy? What is the outside option?

Example: 5 products. Means and Std Dev of Utilities (independent and normally distributed) of Products are

Item	1	2	3	4	5
Mean	6	6.2	6.4	6.6	6.8
Std Dev	2.00000000	2.00000000	2.00000000	2.00000000	1.00000000

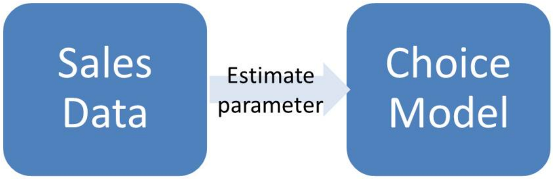
Simulation:

Item	1	2	3	4	5
Probability	0.154666667	0.200667	0.217667	0.240667	0.186333

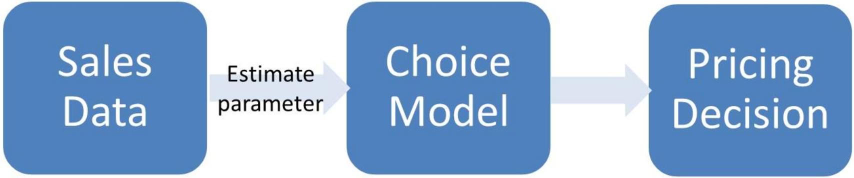
Learning from Experiments

Choose a Parametric Form

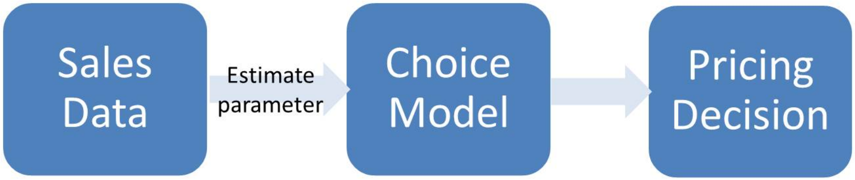
Learning from Experiments



Learning from Experiments



Learning from Experiments



- Heavily depend on "Good guess" of the underline choice model
- Pricing problem is convex only under some well studied choice models(pricing with MNL(Song et al, 2007), Nest-L(Li et al, 2011))
- Parameters Estimation can be extremely complicated (eg. Mixed MNL)

Learning from Experiments

Figure: Can we use experimental data to guid ethe choice of choice model (or Marginal Distribution)?

Approximation: Marginal Distribution Models

Define

$$Z(\tilde{\mathbf{U}}_\mathbf{i}) = \max\left\{\sum_{\mathbf{k}\in\mathcal{K}} \tilde{\mathbf{U}}_{\mathbf{ik}}\mathbf{y}_{\mathbf{ik}} : \sum_{\mathbf{k}\in\mathcal{K}} \mathbf{y}_{\mathbf{ik}} = \mathbf{1},\ \mathbf{y}_{\mathbf{ik}} \in \{\mathbf{0},\mathbf{1}\}\ \forall \mathbf{k} \in \mathcal{K}\right\}.$$

Solve

$$\max_{\theta\in\Theta} E_\theta\left(Z(\tilde{\mathbf{U}}_\mathbf{i})\right).$$

When Θ denotes the family of probability distributions with prescribed marginals, we obtain the **Marginal Distribution Model (MDM)**.

Approximation: Marginal Distribution Models

Define

$$Z(\tilde{\mathbf{U}}_{\mathbf{i}}) = \max\left\{\sum_{\mathbf{k}\in\mathcal{K}} \tilde{\mathbf{U}}_{\mathbf{ik}}\mathbf{y}_{\mathbf{ik}} : \sum_{\mathbf{k}\in\mathcal{K}} \mathbf{y}_{\mathbf{ik}} = \mathbf{1},\ \mathbf{y}_{\mathbf{ik}} \in \{\mathbf{0},\mathbf{1}\}\ \forall \mathbf{k} \in \mathcal{K}\right\}.$$

Solve

$$\max_{\theta\in\Theta} E_\theta\left(Z(\tilde{\mathbf{U}}_{\mathbf{i}})\right).$$

When Θ denotes the family of probability distributions with prescribed marginals, we obtain the **Marginal Distribution Model (MDM)**.

Predict choices using the extremal distribution given by:

$$\theta^* = \arg\max_{\theta\in\Theta} E_\theta\left(Z(\tilde{\mathbf{U}}_{\mathbf{i}})\right).$$

Approximation: Marginal Distribution Models

Theorem (Natarajan, Song and Teo (2009))

For consumer i, assume that the marginal distribution $F_{ik}(.)$ of the error term $\tilde{\epsilon}_{ik}$ is a continuous distribution for all $k \in \mathcal{K}$.

Approximation: Marginal Distribution Models

Theorem (Natarajan, Song and Teo (2009))

For consumer i, assume that the marginal distribution $F_{ik}(.)$ of the error term $\tilde{\epsilon}_{ik}$ is a continuous distribution for all $k \in \mathcal{K}$.

The following concave maximization problem solves the Marginal Distribution Model problem:

$$\max_{\mathbf{P_i}} \left\{ \sum_{k \in \mathcal{K}} \left(V_{ik} P_{ik} + \int_{1-P_{ik}}^{1} F_{ik}^{-1}(t)dt \right) : \sum_{k \in \mathcal{K}} P_{ik} = 1,\ P_{ik} \geq 0\ \forall k \in \mathcal{K}$$

and the choice probabilities under an extremal distribution θ^* is the optimal solution vector $\mathbf{P_i^*}$.

We can solve a compact convex programming problem to obtain the choice probabilities for the extremal distribution.

Marginal Distribution Models

First Order Conditions are necessary and sufficient:

$$P^*_{ik} = 1 - F_{ik}(\lambda_i - V_{ik}), \tag{2}$$

where the Lagrange multiplier λ_i satisfies the following normalization condition:

$$\sum_{k \in \mathcal{K}} P^*_{ik} = \sum_{k \in \mathcal{K}} \Big(1 - F_{ik}(\lambda_i - V_{ik})\Big) = 1. \tag{3}$$

Marginal Distribution Models

First Order Conditions are necessary and sufficient:

$$P_{ik}^* = 1 - F_{ik}(\lambda_i - V_{ik}), \tag{2}$$

where the Lagrange multiplier λ_i satisfies the following normalization condition:

$$\sum_{k \in \mathcal{K}} P_{ik}^* = \sum_{k \in \mathcal{K}} \Big(1 - F_{ik}(\lambda_i - V_{ik}) \Big) = 1. \tag{3}$$

Suppose $F_{ik}(\epsilon) = 1 - e^{-\epsilon}$ for $\epsilon \geq 0$.
Solving the FOC:

$$P_{ik} = \frac{e^{V_{ik}}}{\sum_{l \in \mathcal{K}} e^{V_{il}}} = LOGIT!$$

Marginal Distribution Models

Theorem (MIshra et al. (2014))

Set $V_{i1} = 0$. *Assume MDM with error terms* $\tilde{\epsilon}_{ik}, k \in \mathcal{K}$ *that have a strictly increasing continuous marginal distribution* $F_{ik}(\cdot)$ *defined either on a semi-infinite support* $[\underline{\epsilon}_{ik}, \infty)$ *or an infinite support* $(-\infty, \infty)$. *Let* Δ_{K-1} *be the* $K-1$ *dimensional simplex of choice probabilities:*

$$\Delta_{K-1} = \left\{ \mathbf{P_i} = (\mathbf{P_{i1}}, \ldots, \mathbf{P_{iK}}) : \sum_{\mathbf{k} \in \mathcal{K}} \mathbf{P_{ik}} = \mathbf{1},\ \mathbf{P_{ik}} \geq \mathbf{0}\ \forall \mathbf{k} \in \mathcal{K} \right\}.$$

Let $\Phi(V_{i2}, \ldots, V_{iK}) : \Re_{K-1} \to \Delta_{K-1}$ *be a mapping from the deterministic components of the utilities to the choice probabilities under MDM. Then* ϕ *is a bijection between* $\Re_{K-1}$ *and the interior of the simplex* Δ_{K-1}.

MDM can model almost any choice formula!

Change of variables to $x_i(\tilde{U})$:

$$\begin{aligned} \sup_{\theta \in \Theta} \mathbb{E} \quad [\max_{\mathbf{x}} \quad & \sum_{i=0}^{N} (\tilde{U}_i - p_i) x_i(\tilde{\mathbf{U}})] \\ s.t. \quad & \sum_{i=0}^{N} x_i(\tilde{\mathbf{U}}) = 1 \\ & x_i \in \{0, 1\} \quad i = 0, 1, \ldots, N. \end{aligned} \tag{4}$$

Change of variables to $x_i(\tilde{U})$:

$$\begin{aligned} \sup_{\theta \in \Theta} \mathbb{E} \quad [\max_{\mathbf{x}} \quad & \sum_{i=0}^{N} (\tilde{U}_i - p_i) x_i(\tilde{\mathbf{U}})] \\ s.t. \quad & \sum_{i=0}^{N} x_i(\tilde{\mathbf{U}}) = 1 \\ & x_i \in \{0,1\} \quad i = 0, 1, \ldots, N. \end{aligned} \tag{4}$$

Obtained from solving the following concave maximization problem:

$$\begin{aligned} \max \quad & -\sum_{i=1}^{N} p_i x_i + \sum_{i=0}^{N} \int_{1-x_i}^{1} F_i^{-1}(t) dt \\ s.t. \quad & \sum_{i=0}^{N} x_i = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{5}$$

Optimality condition on (5) yields

$$p_i = F_i^{-1}(1 - x_i) - F_0^{-1}(1 - x_0), \quad \sum_{i=0}^{N} x_i = 1.$$

Optimality condition on (5) yields

$$p_i = F_i^{-1}(1 - x_i) - F_0^{-1}(1 - x_0), \quad \sum_{i=0}^{N} x_i = 1.$$

Projecting out the variables **p**:

$$\begin{aligned}
\max_{\mathbf{x}\geq\mathbf{0}} \quad & -\sum_{i=1}^{N} w_i x_i + \sum_{i=1}^{N} [x_i F_i^{-1}(1 - x_i)] - (1 - x_0) F_0^{-1}(1 - x_0) \\
s.t. \quad & \sum_{i=0}^{N} x_i = 1 \\
& x_i \geq 0, \quad i = 0, 1, \ldots, N
\end{aligned} \tag{6}$$

- Define

$$F_i(\epsilon) = 1 - e^{-\epsilon}, \epsilon \geq 0, i = 0, \ldots, N$$

recover the pricing optimization model in Song et al, 2007 for LOGIT

- Define

$$F_{ik}(\epsilon) = 1 - e^{-\epsilon}\left(\sum_{j=1}^{M_k} e^{a_{jk} - b_k p_{jk}}\right)^{\tau_k - 1}, \ \epsilon \geq (\tau_k - 1)\ln\left(\sum_{j=1}^{M_k} e^{a_{jk} - b_k p_{jk}}\right)$$

recover the pricing model in Li et al, 2011 for Nested LOGIT.

- Define
$$F_i(\epsilon) = 1 - e^{-\epsilon}, \epsilon \geq 0, i = 0, \ldots, N$$
recover the pricing optimization model in Song et al, 2007 for LOGIT
- Define
$$F_{ik}(\epsilon) = 1 - e^{-\epsilon}(\sum_{j=1}^{M_k} e^{a_{jk} - b_k p_{jk}})^{\tau_k - 1}, \ \epsilon \geq (\tau_k - 1)\ln(\sum_{j=1}^{M_k} e^{a_{jk} - b_k p_{jk}})$$
recover the pricing model in Li et al, 2011 for Nested LOGIT.

Both are shown to be convex with respect to market share.

Generalization:

Theorem 2

Under Conditions

A1. The marginal distribution of each product $F_i, i = 1, \ldots, N$ satisfied that $xF_i^{-1}(1-x), i = 1, \ldots, N$ is concave function.

A2. The distribution of outside option F_0 satisfied that $xF_0^{-1}(x)$ is convex function.

the optimal pricing problem (8) is a convex problem with respect to market share $\mathbf{x}$. If the optimal solution of (8) is $\mathbf{x}^*$, then the optimal price strategy is

$$p_i^* = F_i^{-1}(1-x_i^*) - F_0^{-1}(1-x_0^*), i = 1, 2, \ldots, N.$$

Proposition 1

Let $F(x)$ is cumulative distribution function, then

(i) Function $xF^{-1}(1-x)$ is concave if and only if function $\frac{1}{1-F(x)}$ is convex.

(ii) Function $xF^{-1}(x)$ is convex if and only if function $\frac{1}{F(x)}$ is convex.

Proposition 1

Let $F(x)$ is cumulative distribution function, then

(i) Function $xF^{-1}(1-x)$ is concave if and only if function $\frac{1}{1-F(x)}$ is convex.

(ii) Function $xF^{-1}(x)$ is convex if and only if function $\frac{1}{F(x)}$ is convex.

Corollary 1

Condition A1 and A2 hold if the marginal distributions satisfy the following conditions:(i) The tail distribution $\bar{F}_i(y), i = 1, \ldots, N$ is log-concave; (ii) The distribution $F_0(y)$ is log-concave.

How to optimize the prices given a sales data set?

How to optimize the prices given a sales data set?

Recall:

$$\max_{\mathbf{x}\geq\mathbf{0}} \quad -\sum_{i=1}^{N} w_i x_i + \sum_{i=1}^{N}[x_i F_i^{-1}(1-x_i)] - (1-x_0)F_0^{-1}(1-x_0)$$
$$s.t. \quad \sum_{i=0}^{N} x_i = 1$$
$$x_i \geq 0, \quad i = 0, 1, \ldots, N \tag{7}$$

How to optimize the prices given a sales data set?

Recall:

$$\begin{aligned} \max_{\mathbf{x}\geq \mathbf{0}} \quad & -\sum_{i=1}^{N} w_i x_i + \sum_{i=1}^{N} [x_i F_i^{-1}(1-x_i)] - (1-x_0)F_0^{-1}(1-x_0) \\ s.t. \quad & \sum_{i=0}^{N} x_i = 1 \\ & x_i \geq 0, \quad i = 0, 1, \ldots, N \end{aligned} \tag{7}$$

Theorem 2

Convexity Preserving Conditions

A1. The marginal distribution of each product $F_i, i = 1, \ldots, N$ satisfied that $xF_i^{-1}(1-x), i = 1, \ldots, N$ is concave function.

A2. The distribution of outside option F_0 satisfied that $xF_0^{-1}(x)$ is convex function.

Choose $F_i(\cdot)$ to satisfy these properties!

How to optimize the prices given a sales data set?

Recall:

$$\max_{\mathbf{x}\geq\mathbf{0}} \quad -\sum_{i=1}^{N} w_i x_i + \sum_{i=1}^{N}[x_i F_i^{-1}(1-x_i)] - (1-x_0)F_0^{-1}(1-x_0)$$

$$s.t. \quad \sum_{i=0}^{N} x_i = 1$$

$$x_i \geq 0, \quad i = 0,1,\ldots,N \tag{8}$$

Define

$$y_{0k} := (1-x_{0k})F_0^{-1}(1-x_{0k})$$

$$y_{ik} := x_{ik}F_i^{-1}(1-x_{ik}), i = 1,\ldots,N, k = 1,\ldots,M$$

Estimation Problem

Penalize deviation from FOC, while preserving convexity and monotinicty condition for marginals

$$\min_{y_{i,k}} \sum_{i=1}^{N} \sum_{k=1}^{M} \frac{\left| \frac{y_{i,k}}{x_{i,k}} - \frac{y_{0,k(i)}}{1-x_{0,k(i)}} - p_{i,k(i)} \right|}{p_{i,k(i)}} \tag{9}$$

$$s.t. \frac{x_{i,k} - x_{i,k-1}}{x_{i,k+1} - x_{i,k-1}} y_{i,k+1} + \frac{x_{i,k+1} - x_{i,k}}{x_{i,k+1} - x_{i,k-1}} y_{i,k-1} \leq y_{i,k}, \forall i, k \tag{10}$$

$$\frac{x_{0,k} - x_{0,k-1}}{x_{0,k+1} - x_{0,k-1}} y_{0,k+1} + \frac{x_{0,k+1} - x_{0,k}}{x_{0,k+1} - x_{0,k-1}} y_{0,k-1} \geq y_{0,k}, \forall k \tag{11}$$

$$\frac{y_{i,k}}{x_{i,k}} \leq \frac{y_{i,k-1}}{x_{i,k-1}} \quad \forall i, k \tag{12}$$

$$\frac{y_{0,k}}{1 - x_{0,k}} \leq \frac{y_{0,k-1}}{1 - x_{0,k-1}} \quad \forall k \tag{13}$$

Optimization Problem:

Optimize over the piece wise linear extension of the fitted values:

$$
\begin{aligned}
\Pi := \quad \max \quad & -\sum_{i=1}^{N} w_i x_i + \sum_{i=1}^{N} \delta_i - \delta_0 \\
s.t. \quad & \delta_i \leq y_{i,k} + \frac{y_{i,k+1} - y_{i,k}}{x_{i,k+1} - x_{i,k}} (x_i - x_{i,k}), k = 1, \ldots, M, i = 1, \ldots, N \\
& \delta_0 \geq y_{0,k} + \frac{y_{0,k+1} - y_{0,k}}{x_{0,k+1} - x_{0,k}} (x_0 - x_{0,k}), k = 1, \ldots, M \\
& \sum_{i=0}^{N} x_i = 1 \\
& x_i \leq x_{i,M}, \forall i = 0, ..., N \\
& x_i \geq x_{i,1}, \forall i = 0, ..., N \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
\tag{14}
$$

Proposition 2

For those model whose underline marginal distribution satisfy Proposition 1, the estimation based optimization method converge to the true optimal price when number of experiments goes to infinity.

This approach can also be used to calibrate parametric choice models based on MLE.

Back to motivating example

Table: Prices Comparison

Product	Current Price	Proposed price
1	45520.91399	41018.1861
2	46906.03611	44608.3848
3	45056.02187	40606.392
4	43874.26176	41728.8295
5	47028.20538	44727.9726
6	42044.17735	39988.7095
7	45522.53382	41030.0762
8	39455	43412.7395
9	37955	39878.1332
10	33182.13546	34857.0785
11	27255	29989.6299
12	35926.64048	39519.3045
13	33550.38213	36905.4203
14	37955	39874.9055
15	39750	39749.5152
16	[illegible]	[illegible]

Back to motivating example

7.19% improvement

Table: Prices Comparison

Product	Current Price	Proposed price
1	45520.91399	41018.1861
2	46906.03611	44608.3848
3	45056.02187	40606.392
4	43874.26176	41728.8295
5	47028.20538	44727.9726
6	42044.17735	39988.7095
7	45522.53382	41030.0762
8	39455	43412.7395
9	37955	39878.1332
10	33182.13546	34857.0785
11	27255	29989.6299
12	35926.64048	39519.3045
13	33550.38213	36905.4203
14	37955	39874.9055
15	39750	39749.5152

Table: Attributes and levels

Product	Brand	Engine Capacity	Drive Type	Fuel Economy (mpg)
1	1	2.5	1	28.3
2	1	3	2	20.51
3	1	3	1	21.57
4	1	3.6	2	20.36
5	1	3.6	2	25.4
6	1	3.6	1	21.32
7	1	3.6	1	26.44
8	2	3.6	2	25.4
9	2	2.5	1	28.3
10	2	2.5	1	22.24
11	2	2.5	1	24.88
12	2	3.6	2	20.36
13	2	3.6	1	21.32
14	2	3.6	1	26.44
15	3	2.5	1	28.3
16	3	2.5	1	22.24
17	3	3.6	2	20.36
18	3	3.6	2	25.4
19	3	3.6	1	21.32
20	3	3.6	1	26.44

Assume $\tilde{\beta}$ follows a Multi-normal distribution

$$\tilde{\beta} \sim \left(\begin{pmatrix} -3.23599 \\ 4.47746 \\ 3.76418 \\ 0.891799 \end{pmatrix}, \begin{pmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix} \right)$$

Pricing with Random Coefficient Logit Model

Iteration Procedure

Step 1. Start iteration from the current price, denoted as $\mathbf{p^0}$.

Step 2. Randomly generate a set of prices $\mathbf{p}$. The generated price uniformly distributed with mean $\mathbf{p^0}$ and with deviation $\pm 5\%$ from the base price.

Step 3. Under each price $\mathbf{p}$, we use $x_i^{(k)} = \frac{e^{\frac{\beta_k' \mathbf{x}_i - p_i}{p_i}}}{1 + \sum_{j=1}^{N} e^{\frac{\beta_k' \mathbf{x}_j - p_j}{p_j}}}$ to get the choice probability of Product i under each β_k sampled from the given distribution. Take average of $x_i^{(k)}$ to get Product i's market share X_i under $\mathbf{p}$. Outside market share equals to $1 - \sum_{i=1}^{N} X_i$.

Step 4. Apply proposed procedure to get an optimal price $\mathbf{p}^*$. Then let $\mathbf{p^0} = \mathbf{p}^*$. Go to Step 1.

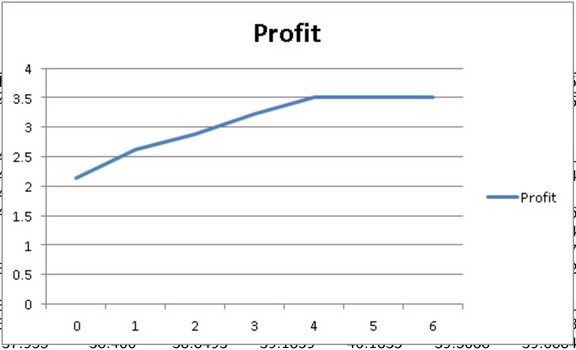
- Generate 10000 samples β_k from the distribution above.
- Price sample size: 500
- Number of product, $N = 20$

- Generate 10000 samples β_k from the distribution above.
- Price sample size: 500
- Number of product, $N = 20$

Table: Optimal prices in the iteration

Product	Round0	Round1	Round2	Round3	Round4	Round5	Round6
1	45.5209	45.2472	43.1932	40.6832	39.1126	39.5322	39.4915
2	46.906	48.1646	60.7988	150.878	—	—	—
3	45.056	46.7618	42.0065	40.431	39.4484	39.4596	38.377
4	43.8743	41.8156	39.3666	38.0697	37.9217	38.352	38.2271
5	47.0282	48.1509	45.4342	42.8126	40.6351	40.7921	40.6864
6	42.0442	42.7233	42.3595	36.4961	36.7091	36.7743	36.774
7	45.5225	42.0075	39.6487	39.329	39.4702	39.7232	39.5706
8	39.455	39.6334	40.2479	41.0651	40.9702	41.1461	40.8434
9	37.955	38.5537	38.1001	38.0518	39.6625	38.3735	39.3487
10	33.1821	33.6556	33.0572	32.3479	33.4629	33.6213	33.3792
11	27.255	28.6387	29.1422	29.624	29.5576	30.0567	29.903
12	35.9266	36.5762	36.9803	37.3772	37.5535	37.9936	37.6681
13	33.5504	34.3787	35.1876	36.3184	35.2446	35.6577	35.4948
14	37.955	38.466	38.8493	39.1859	40.1835	39.5088	39.6884
15	39.75	39.4852	37.8888	37.0477	36.7733	36.0479	36.5594
16	35.809	34.7961	33.8481	32.5897	31.5524	31.517	31.5248
17	38.2274	38.1956	37.7011	36.5208	37.642	37.0075	37.0053
18	41.25	38.5217	39.7071	40.0234	39.5396	40.6001	40.4594
19	36.0612	36.0526	35.186	34.1275	33.9306	35.2729	34.6411
20	39.75	38.0835	37.4906	38.9115	38.7519	40.4837	39.9537
Simulated Profit	2.1378	2.6155	2.8923	3.21788	3.5057	3.5137	3.5146

- Generate 10000 samples β_k from the distribution above.
- Price sample size: 500
- Number of product, $N = 20$



Product							
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							
11							
12							
13							
14							
15	39.75	39.4852	37.8888	37.0477	36.7733	36.0479	36.5594
16	35.809	34.7961	33.8481	32.5897	31.5524	31.517	31.5248
17	38.2274	38.1956	37.7011	36.5208	37.642	37.0075	37.0053
18	41.25	38.5217	39.7071	40.0234	39.5396	40.6001	40.4594
19	36.0612	36.0526	35.186	34.1275	33.9306	35.2729	34.6411
20	39.75	38.0835	37.4906	38.9115	38.7519	40.4837	39.9537
Simulated Profit	2.1378	2.6155	2.8923	3.21788	3.5057	3.5137	3.5146

Concluding Remarks

- Provide a unified model on multi-product pricing with customers' choice model
- Pricing problem remains convex for a large class of MDM choice model
- Data-driven approach for pricing under the random coefficient LOGIT model
- Assortment Problems leads to simple MIP problems.