Approximate Modularity

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Modularity

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$$\arg\max_{S \in \binom{[n]}{k}} f(S) = \{w_{\pi(1)}, \ldots, w_{\pi(k)}\} \quad \left(w_{\pi(1)} \geq \cdots \geq w_{\pi(n)}\right)$$

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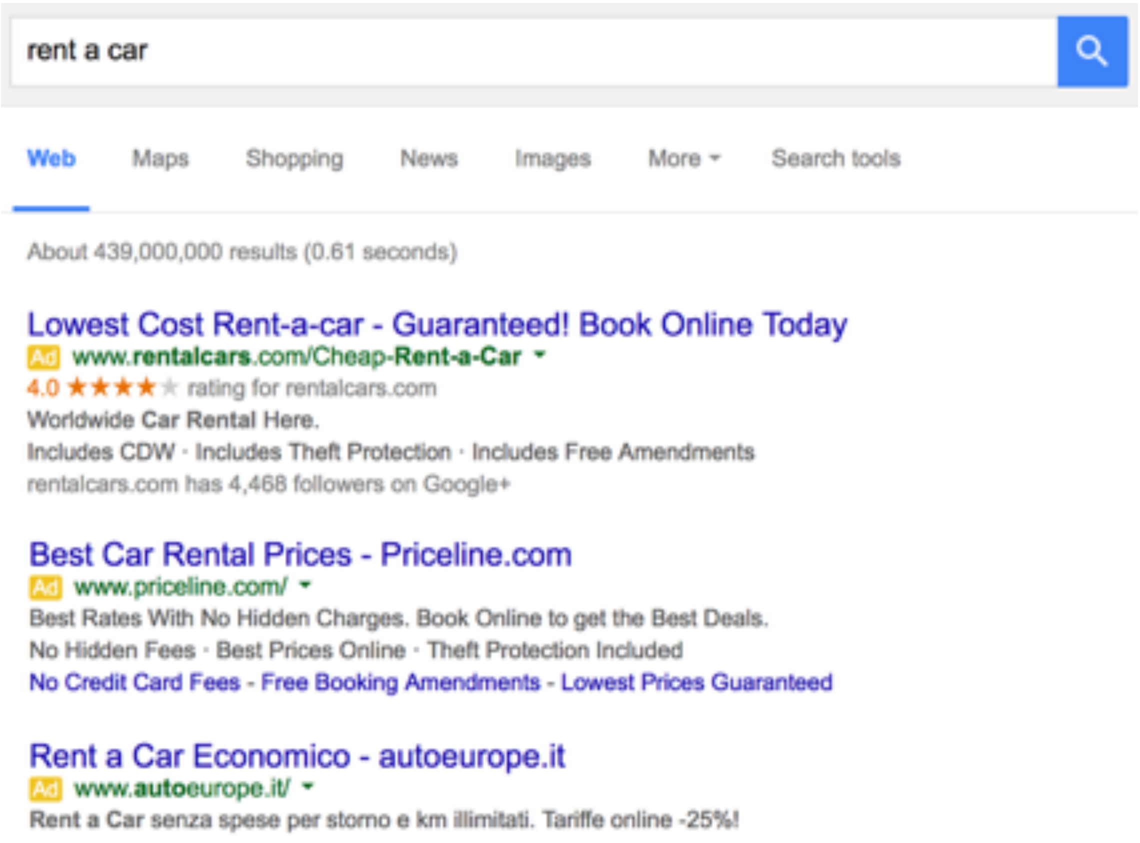
- Modular functions can be succinctly represented as linear functions: $f(S) = w_0 + \sum_{i \in S} w_i$
- It is thus very easy to optimize them $(O(n))$
even though the search space can be huge $\left(O\left(2^n\right)\right)$

An example: Ads



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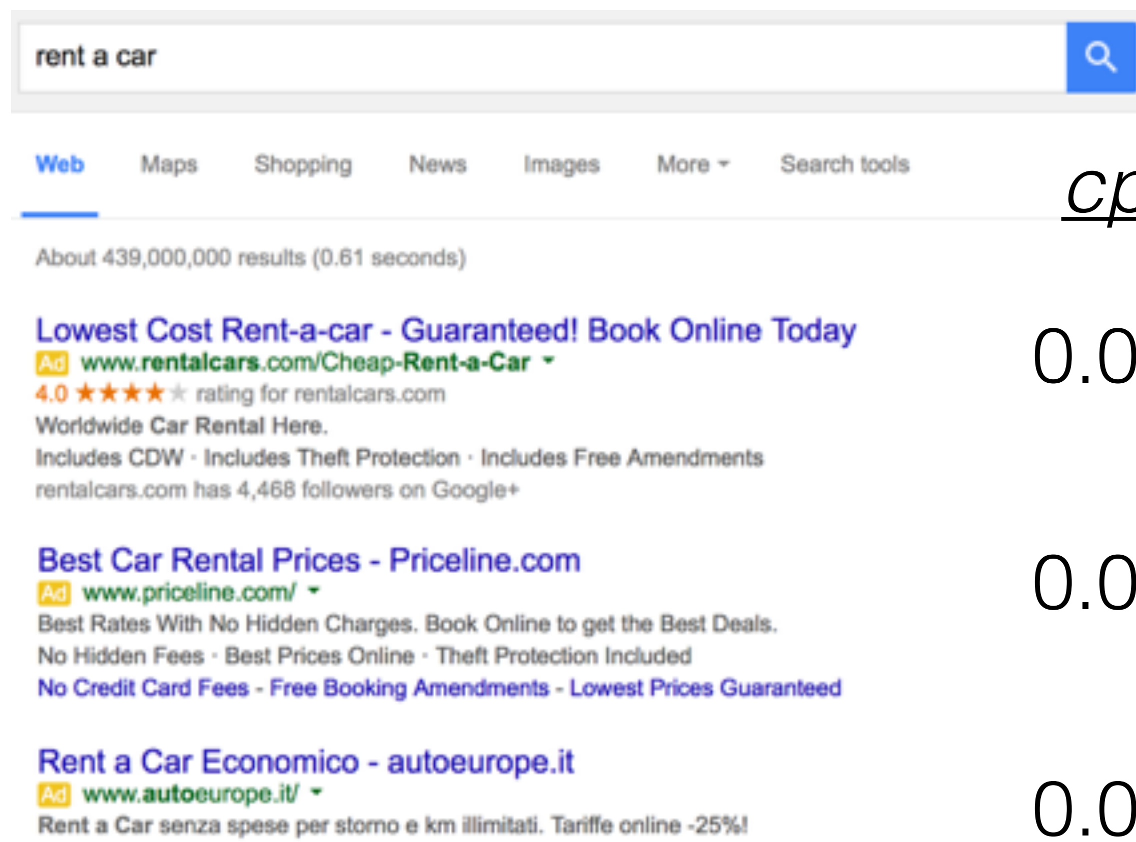


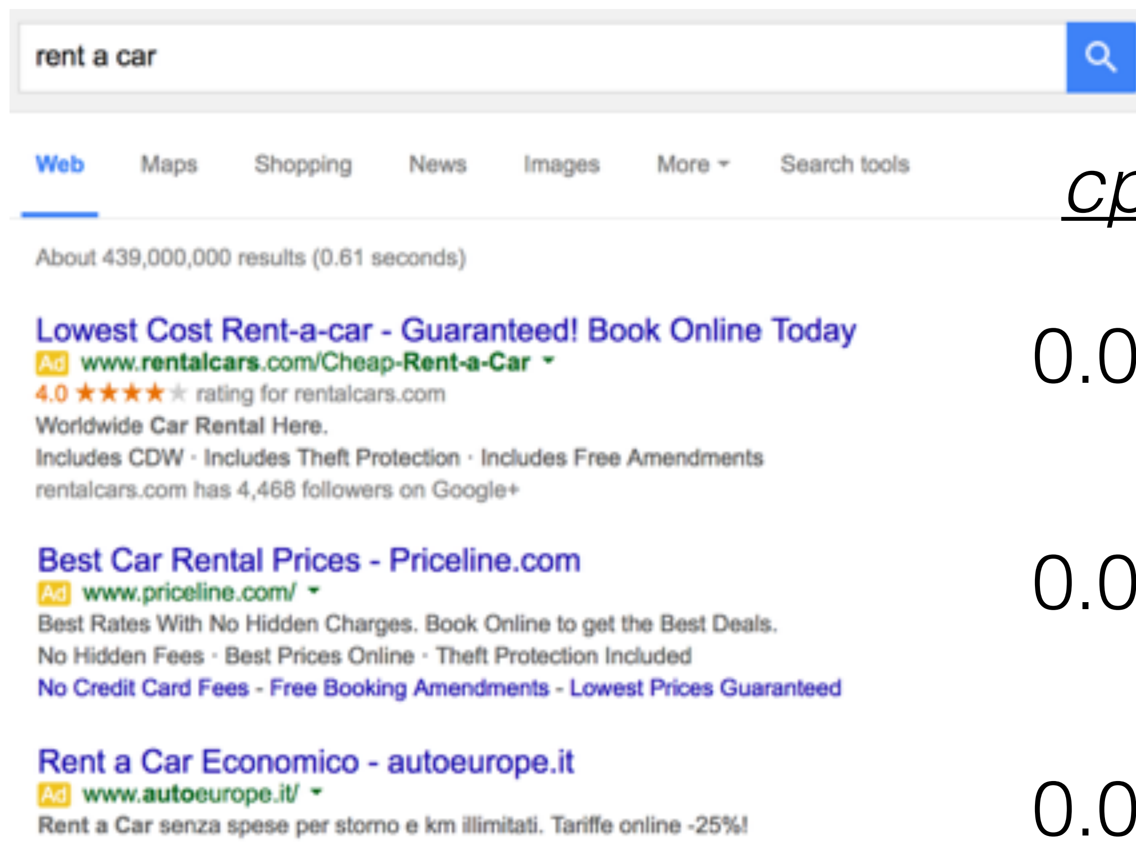
cpc

0.03€

0.05€

0.02€

Independent Clicks Model



cpc	*ctr*
0.03€	**5%**
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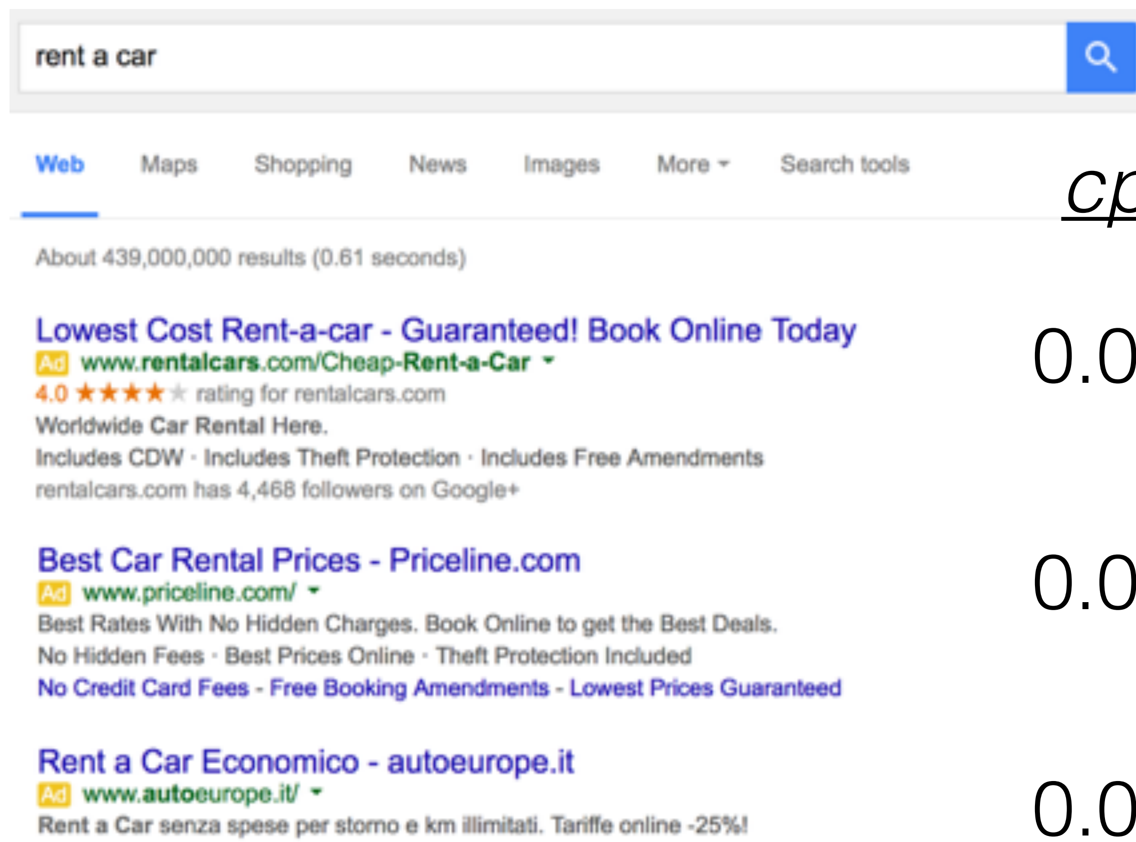

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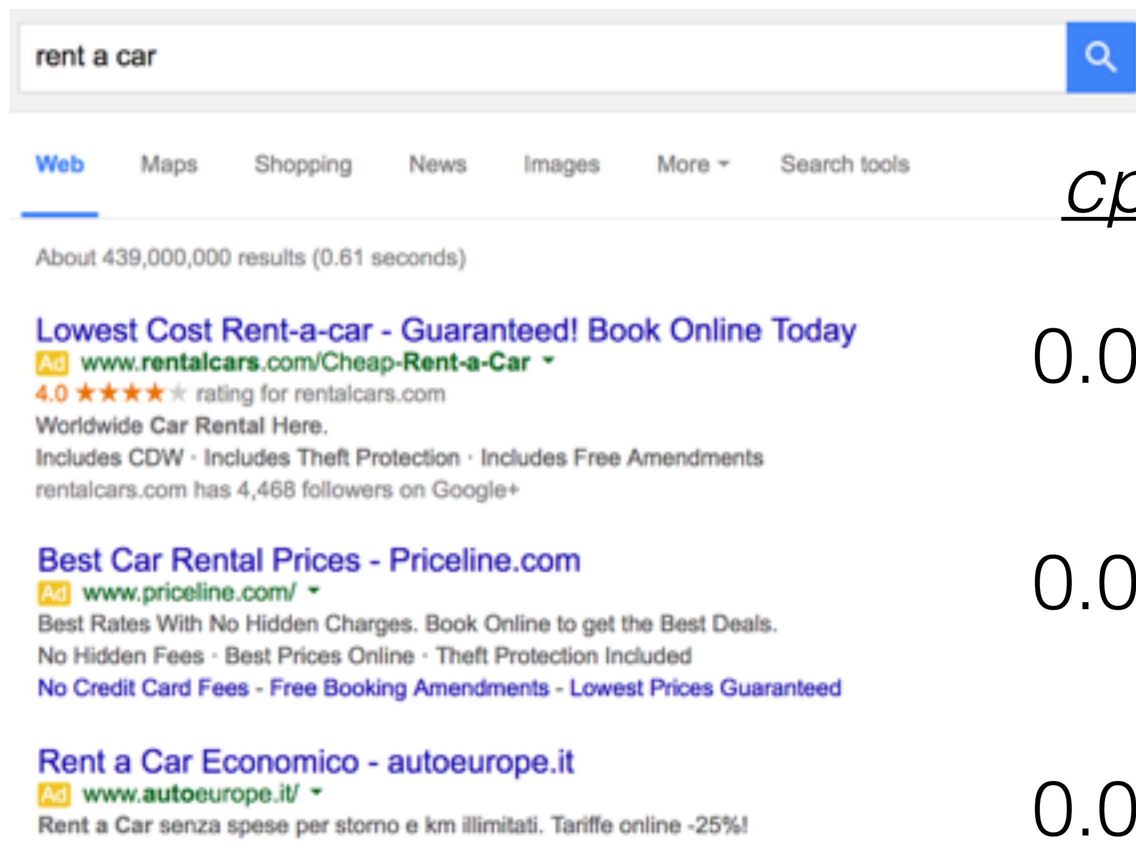

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In practice, clicks are not independent, but…

Approximate Modularity

…*f* satisfies the ϵ-approximate modularity condition

$$f(S) + f(T) = f(S \cup T) + f(S \cap T) \pm \epsilon \qquad \forall S, T \subseteq [n]$$

for some small constant $\epsilon > 0$.

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$$\Big|\big(f(S) + f(T)\big) - \big(f(S \cup T) + f(S \cap T)\big)\Big| \leq \epsilon$$

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In our paper we analyze the *nature*, and the *approximability*, of approximately modular functions

Approximate Properties

The set function f is ϵ-approximately modular if

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Our definition resembles the *approximate linearity* definition of the *Borsuk-Ulam* theory of approximate functions on convex domains

If *D* is a convex domain, *f* is ϵ-approximately linear on *D* if

$$f(x) + f(y) = f(w) + f(z) \pm \epsilon$$

$$\forall x, y, w, z \in D \text{ such that } x + y = w + z$$

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One of the original *B-U* theory's goals is to *approximate* an **approximately linear function** with a **truly linear function**

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- In our case, the domain is composed of the vertices of the hypercube - so it is not closed, nor convex.
  - How well can we optimize these functions in polynomial time?
  - How “close” are these functions to modularity?

Our Results

Polynomial-time Optimization

- Let *f* be an ϵ-approximately modular function on *[n]*. We show that:
  - there exists an algorithm that performs $O(n^2 \log n)$ queries to *f*, and returns a **modular** function *g* such that $\forall S \subseteq [n] \quad |f(S) - g(S)| \leq O(\epsilon \cdot \sqrt{n})$

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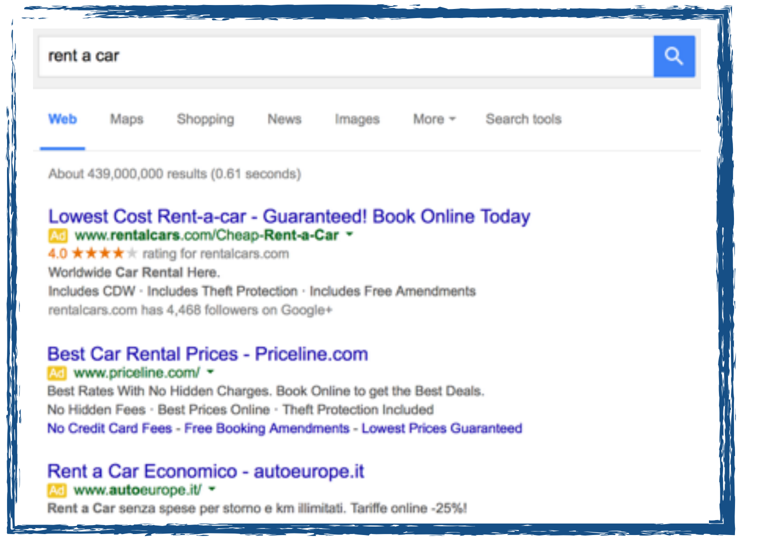
Thus we can, say, approximate our ads problem to an additive $O(\epsilon\sqrt{n})$ error.



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Thus, if we drop the poly-time reconstruction requirement, we can approximate *f* with a modular *g* in an ***exponentially*** better way

Poly-time Approximation

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- That is, our algorithm has to choose $w_0, w_1, \ldots, w_n$, while querying f at most $n^{O(1)}$ times, so to guarantee that $|g(S) - f(S)|$ is small for each $S \subseteq [n]$

Poly-time Approximation

- First, we choose $z = f(\varnothing)$
- Then, we choose:

$$w_i = \text{avg}_{k \in [n]} \text{avg}_{S \in \binom{[n]-\{i\}}{k-1}} \Big(f(S \cup \{i\}) - f(S) \Big)$$

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…but the approximate modularity condition allows us to approximate w_i with only $O(n \log n)$ many random samples

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- With these weights, we can prove that

$$|f(S) - g(S)| \leq 4 \cdot \epsilon \cdot \sqrt{\min(|S|, n - |S|)} \qquad \forall S \subseteq [n]$$

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Observe that the weights, and thus g, are linear combinations of the values of f

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$$\begin{cases} \max g(S) - f(S) \text{ subject to} \\ \forall \{X, Y\} \in \binom{2^{[n]}}{2} : \quad -\epsilon \leq f(X \cup Y) + f(X \cap Y) - f(X) - f(Y) \leq \epsilon \end{cases}$$

Poly-time Approximation

- Our algorithm produces a modular function which approximates *f* to within an additive $O\left(\epsilon \cdot \sqrt{n}\right)$ value with a total of $O\left(n^2 \log n\right)$ queries to *f*.

Inapproximability

- We also show that no algorithm querying an ϵ-approximately modular *f* at most *q* times can distinguish WHP whether:
  - *f* is the constant 0 function; or
  - the maximum value of *f* is at least $\Omega\left(\epsilon \cdot \sqrt{n / \log q}\right)$
- Hence, the approximation of our poly-time algorithm is almost optimal.

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S'

S''

S'''

$\ldots$

sets of f-*minimum* value

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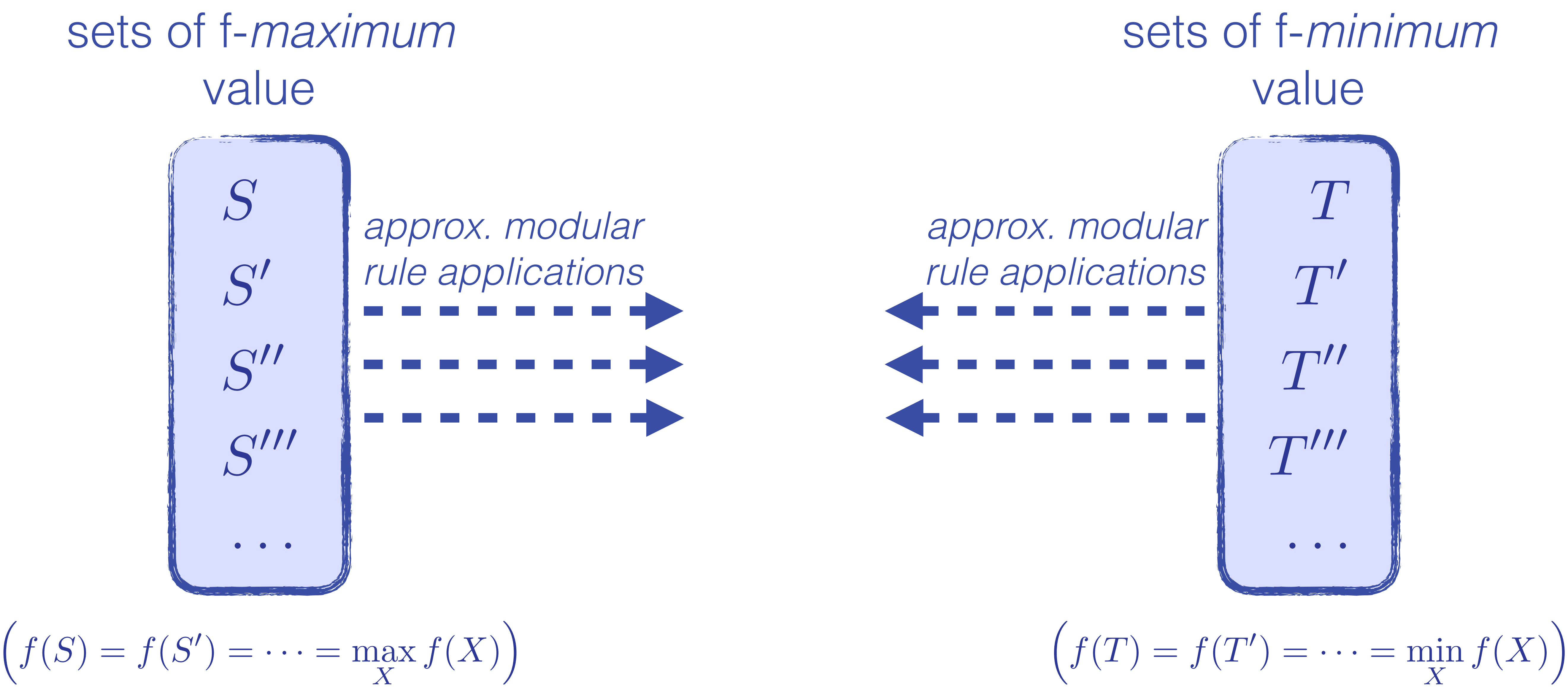

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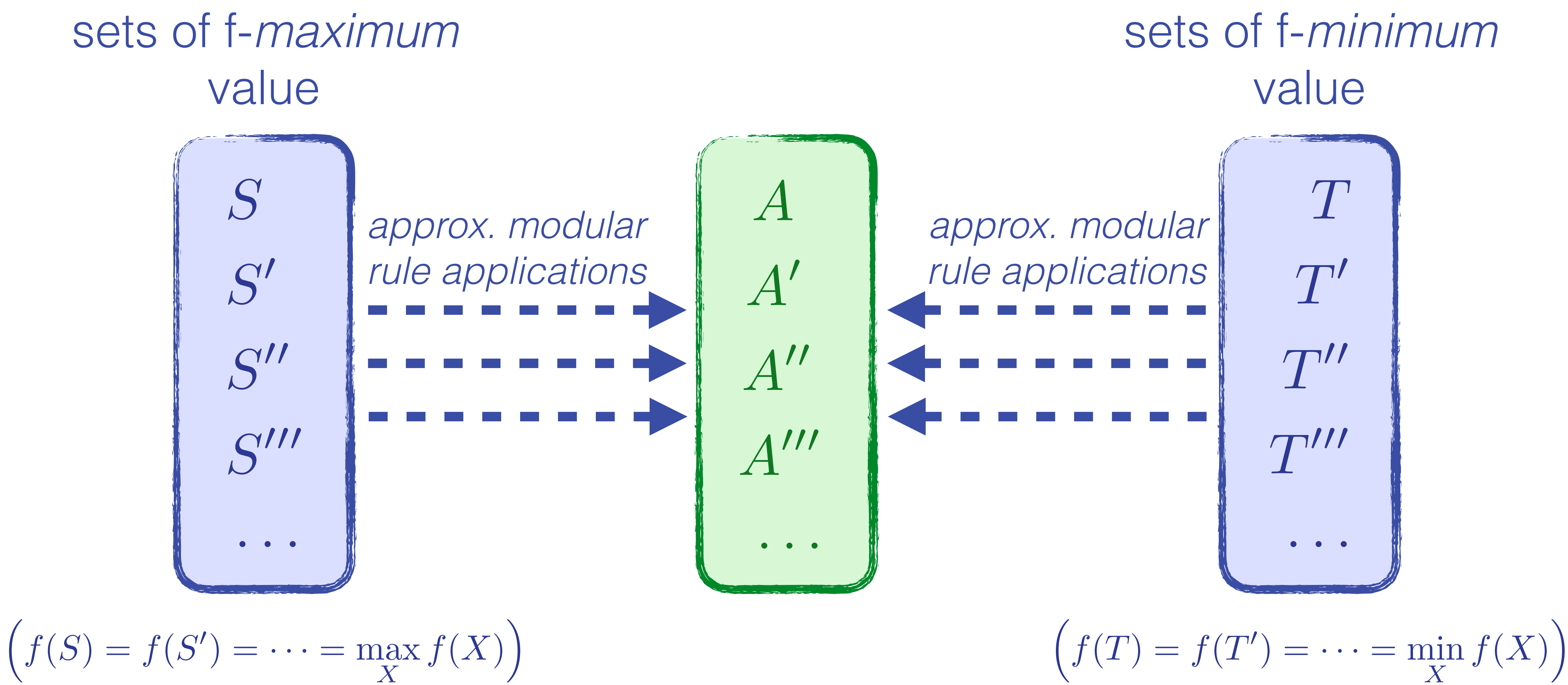

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- Then,
  - the distance from modularity of f equals the distance from modularity of $F = f\text{-}g$
  - the best modular approximation of F is the 0-function z

Thus, bounding the distance from modularity of functions F that are best approximated by z is sufficient

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- Then,
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  - the best modular approximation of *F* is the 0-function *z*
  - the distance from modularity of *F* is equal to the maximum value *M* of *F.*

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P^+: RD on the *F*-maximum sets

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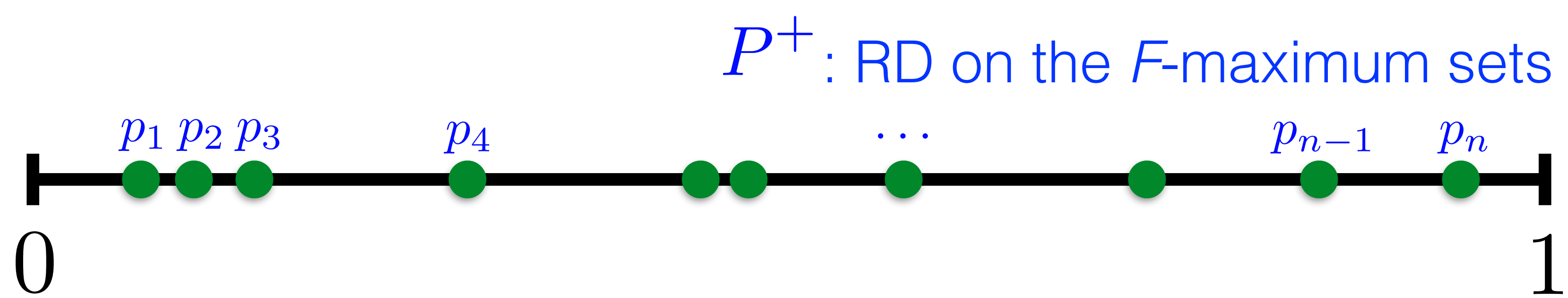

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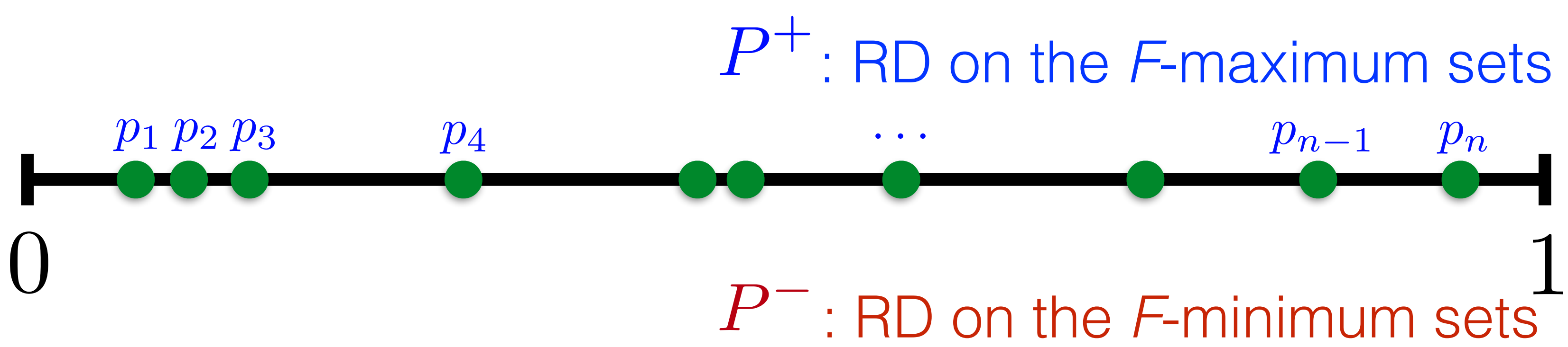

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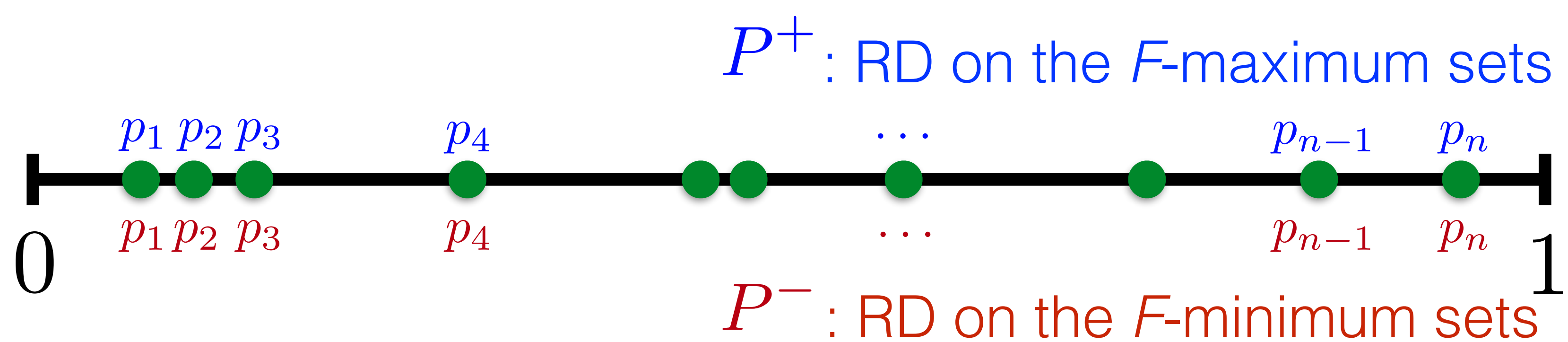
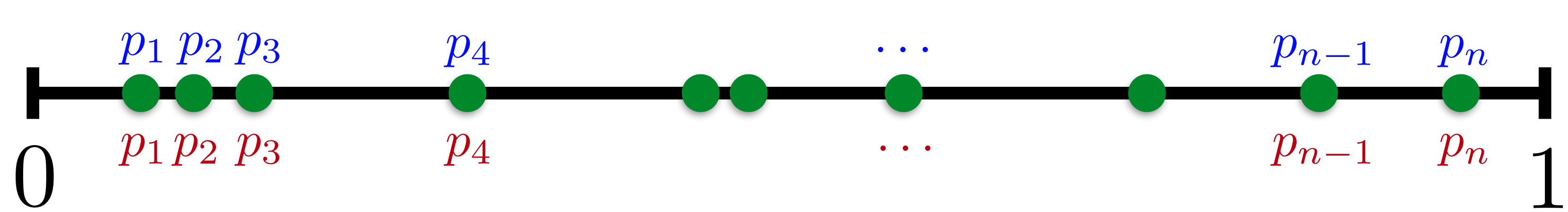


Random Thresholds

- We use the approximately modular rule to create a random system of unions/intersections that, using any random distribution over sets, produces sets that contain all the elements with a marginal (roughly) smaller than a UAR T



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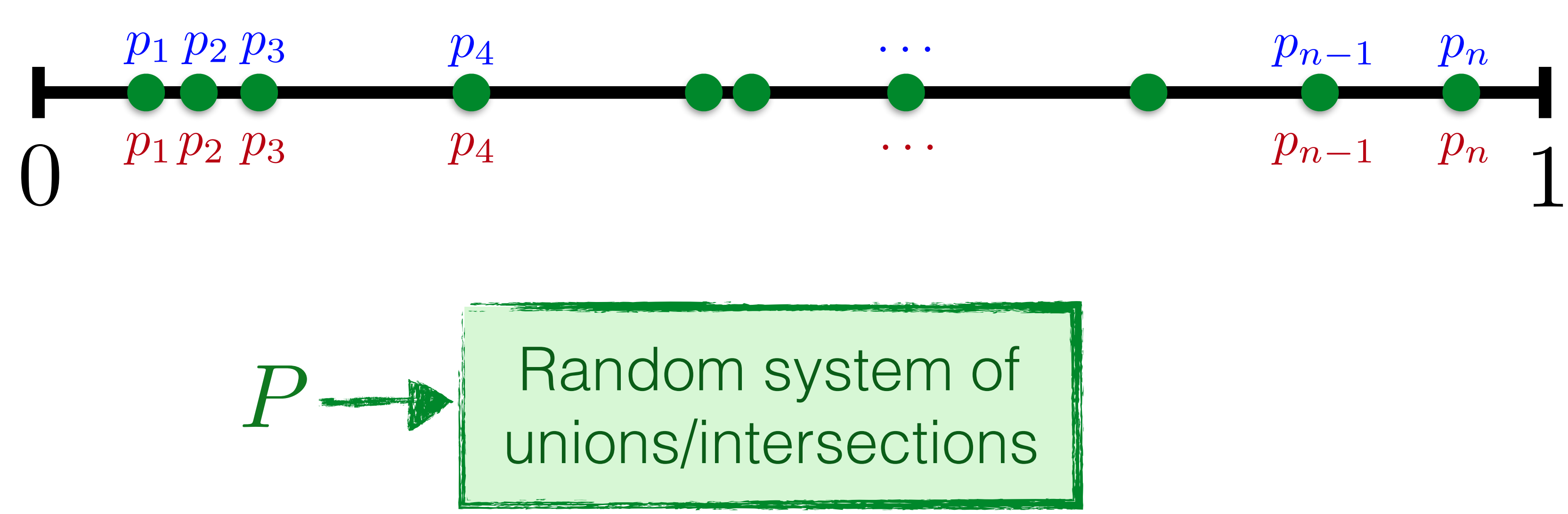

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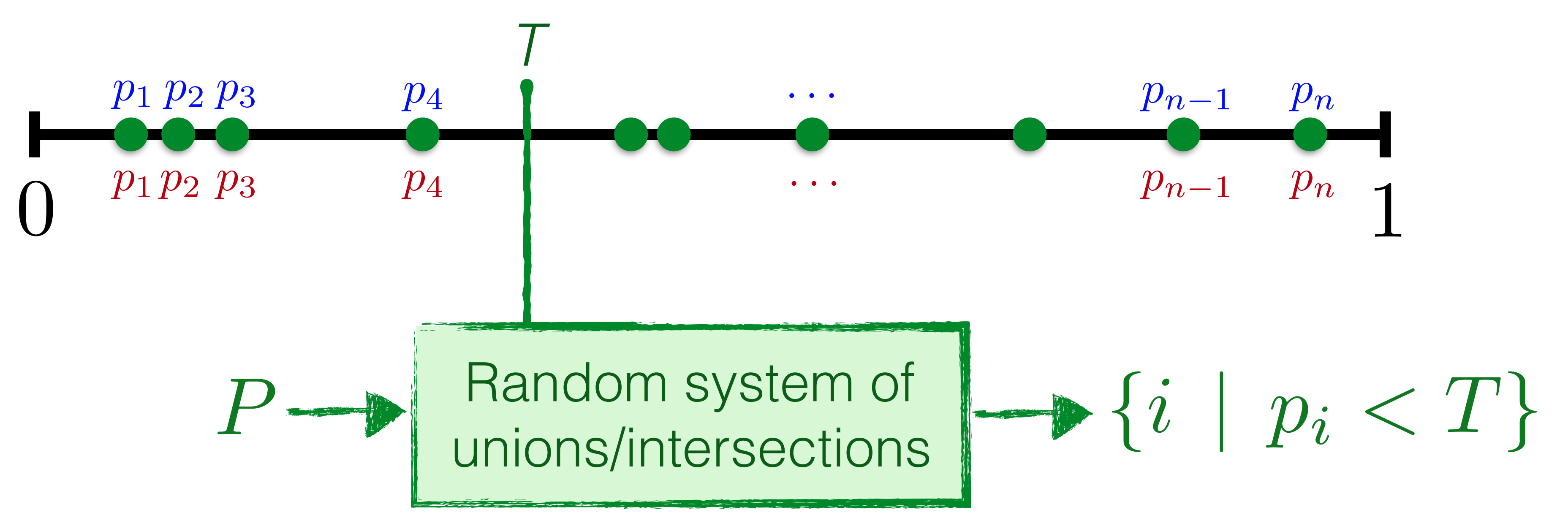

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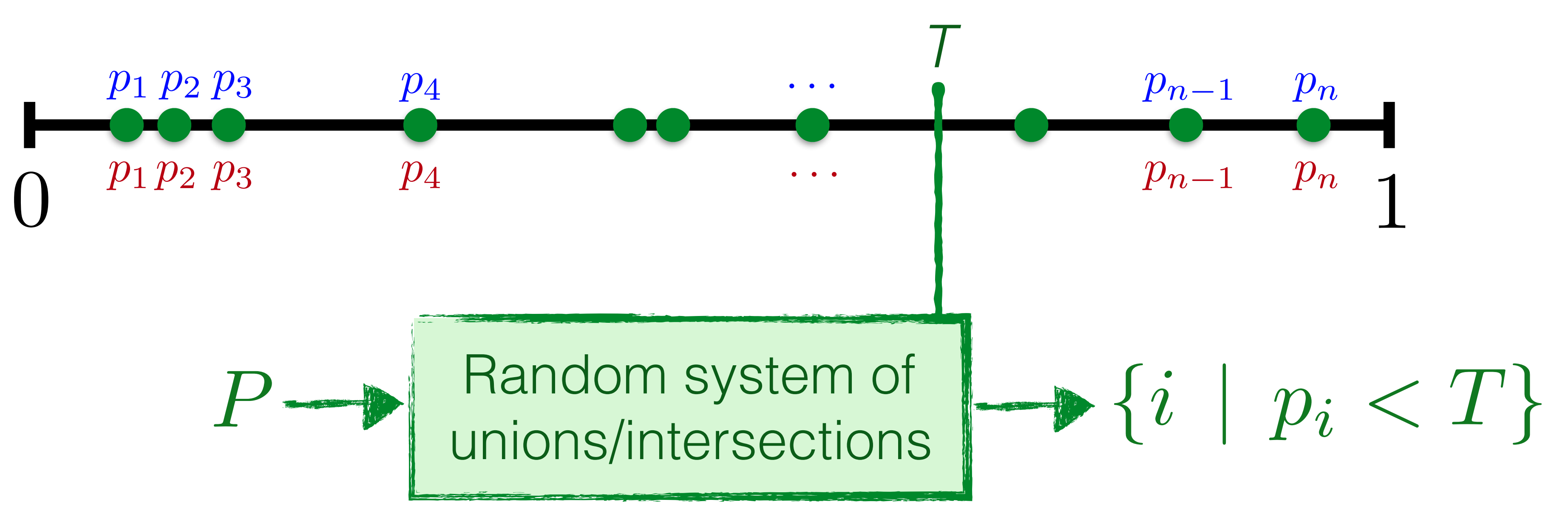

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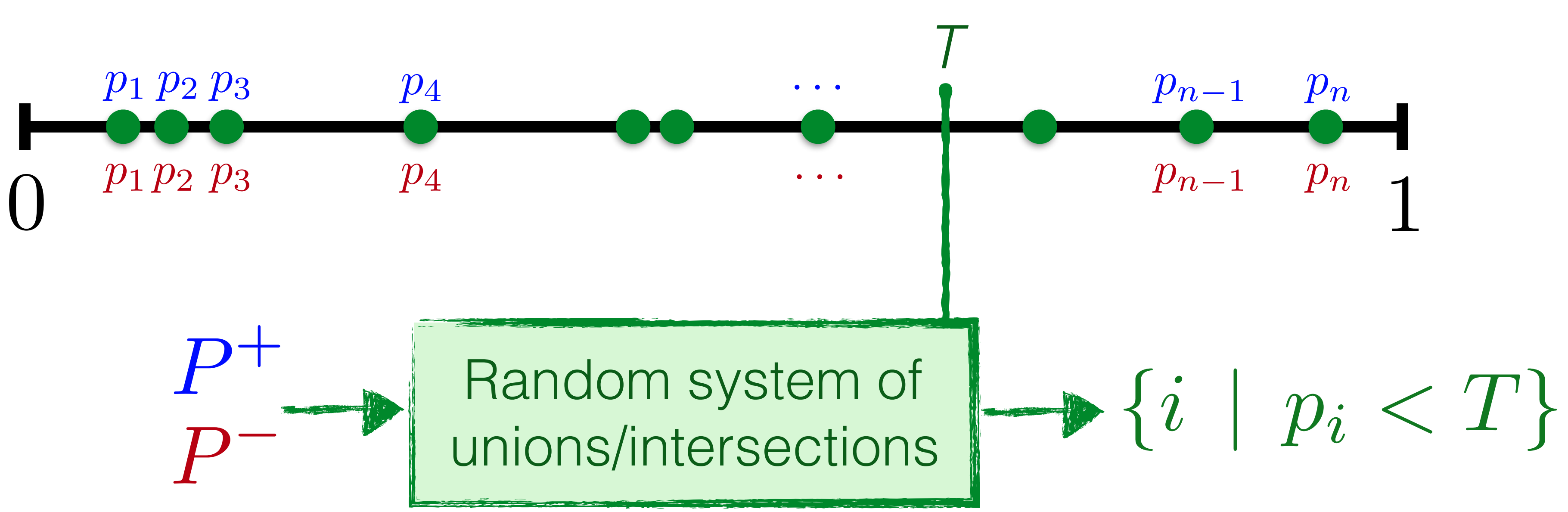
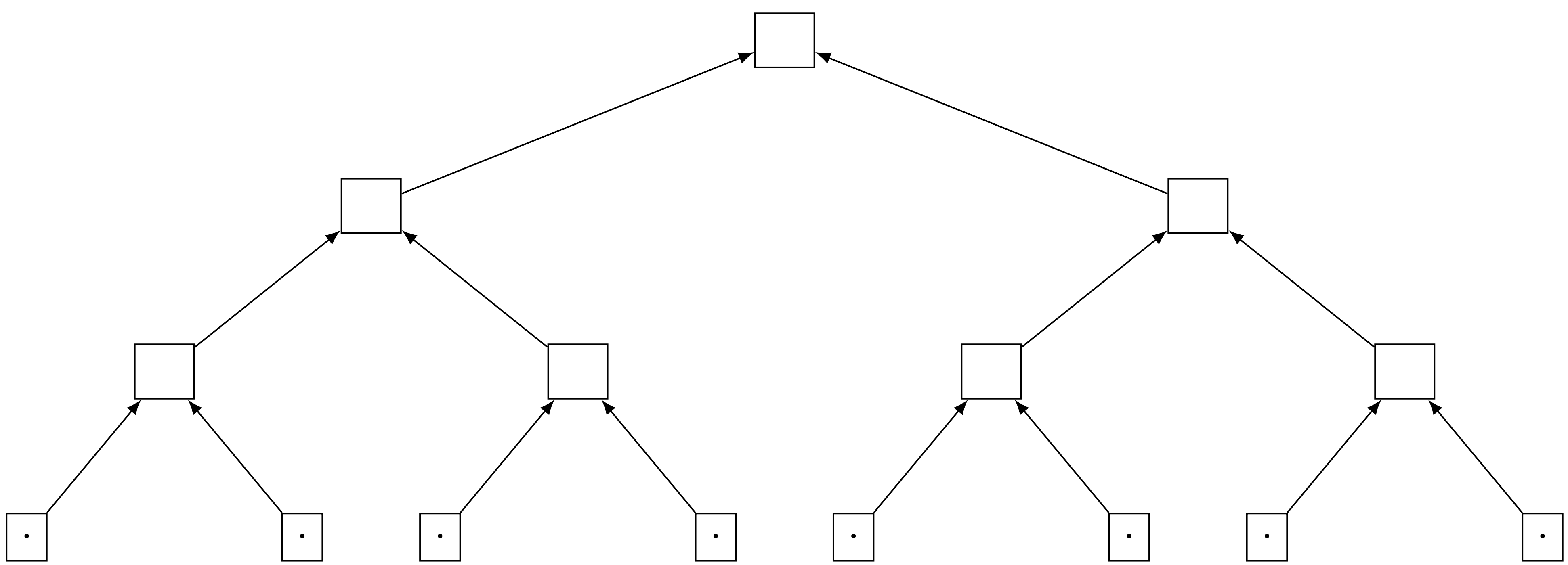
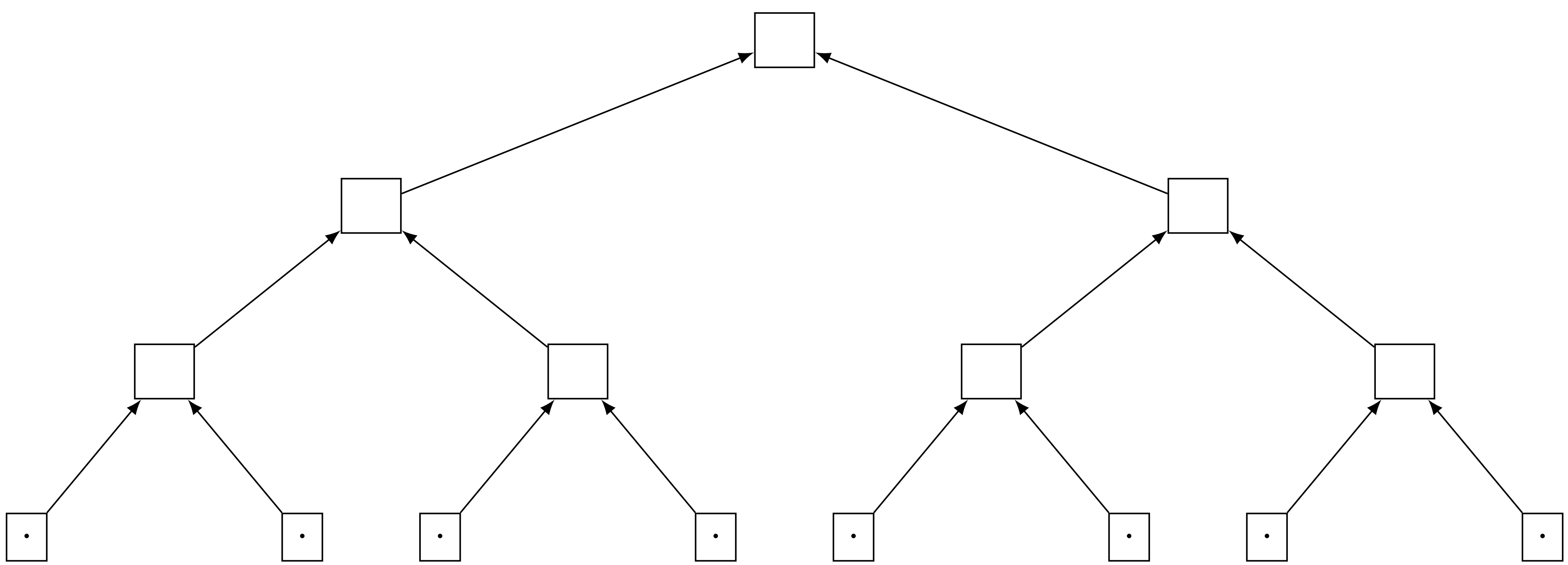


Random System of ∪/∩
Striped Tree

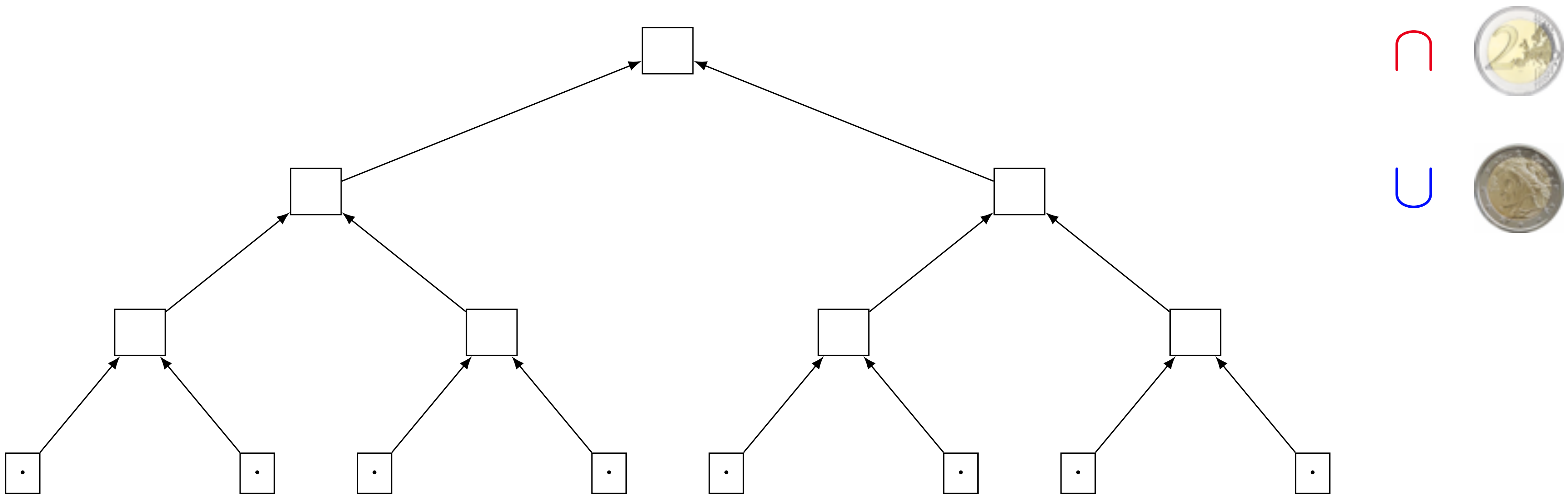


Random System of $\cup/\cap$
Striped Tree

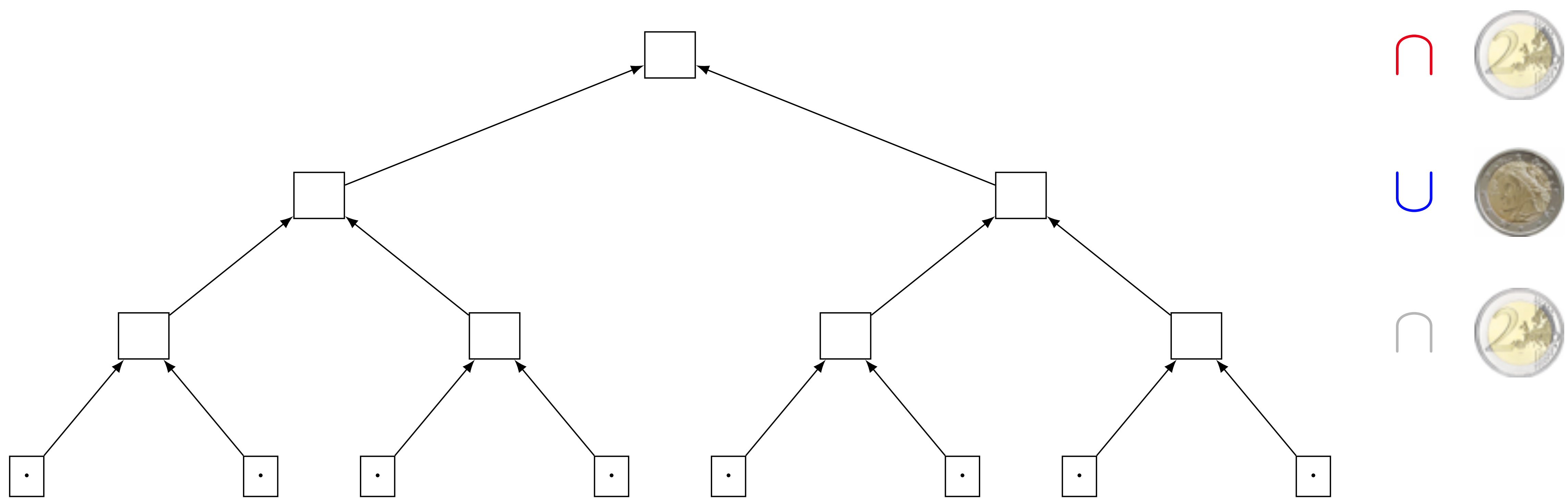


$\cap$

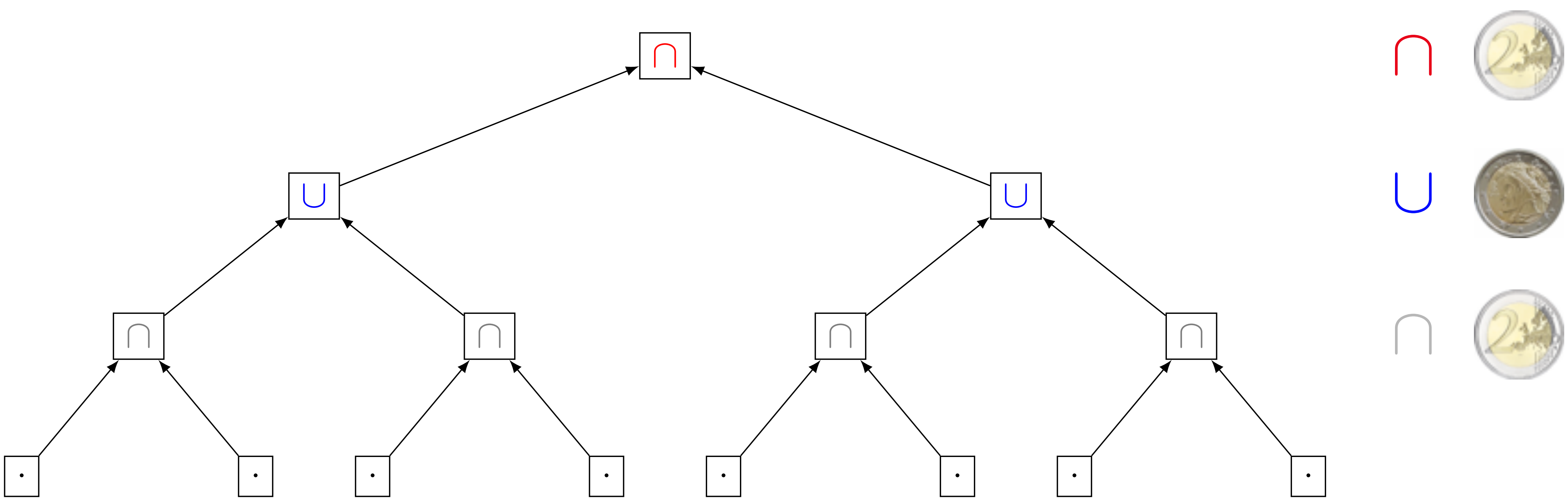
Random System of ∪/∩
Striped Tree



Random System of ∪/∩ Striped Tree



Random System of ∪/∩ Striped Tree



Random System of ∪/∩
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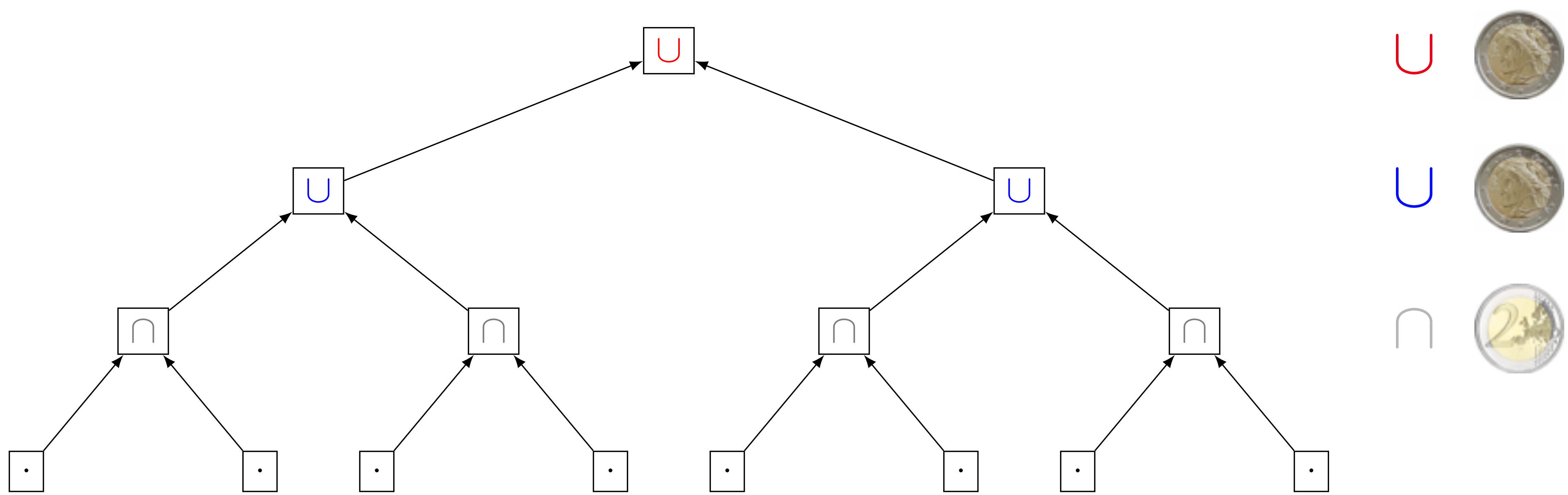

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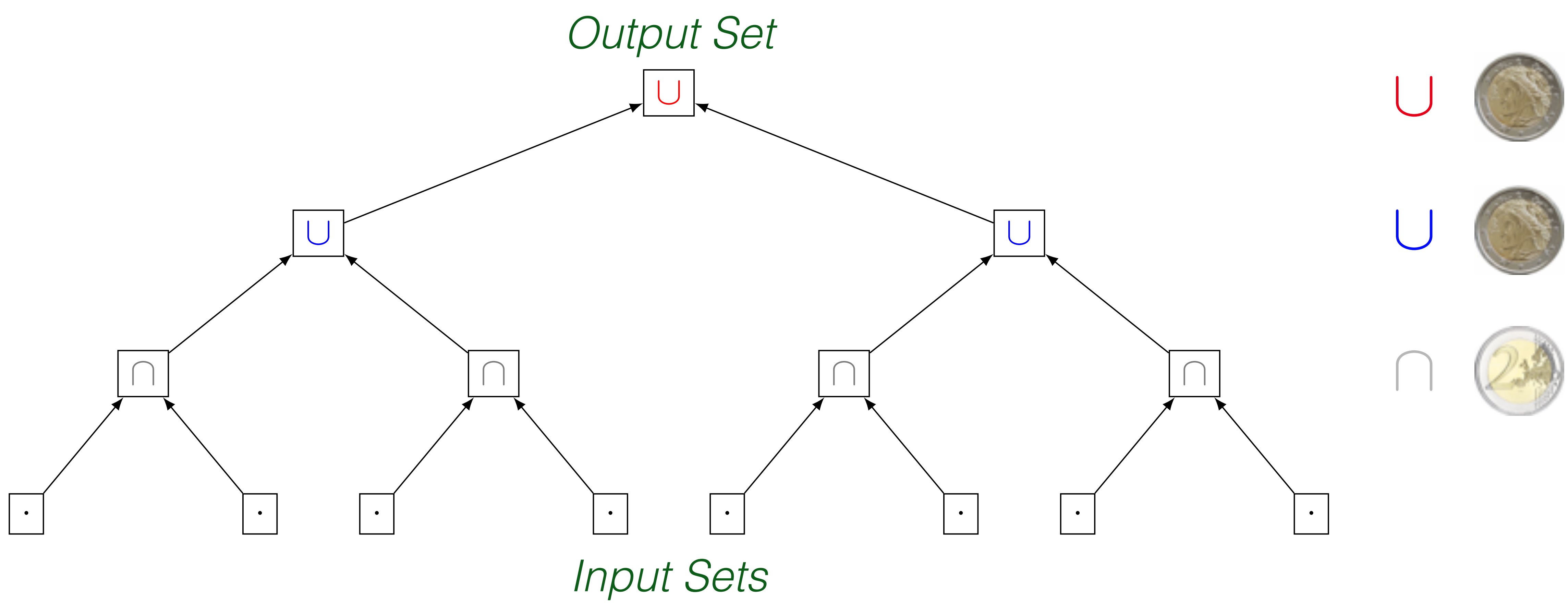
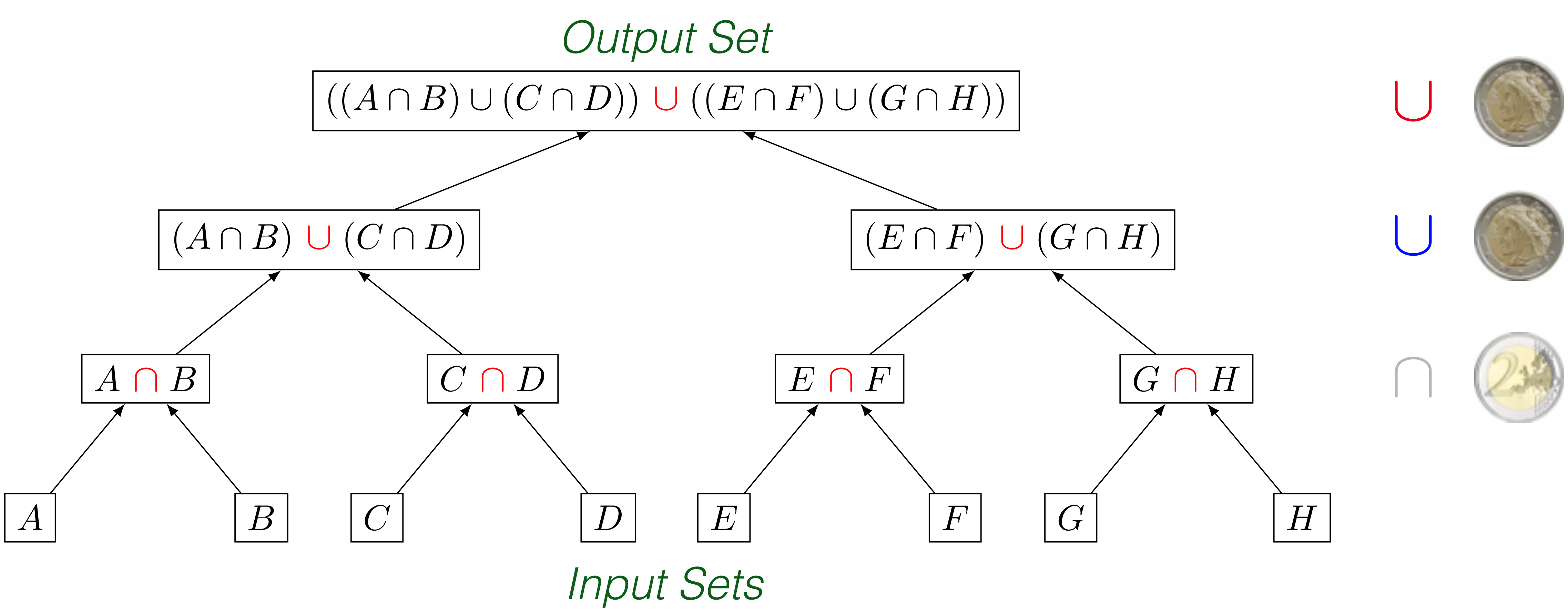


Random System of ∪/∩ Striped Tree



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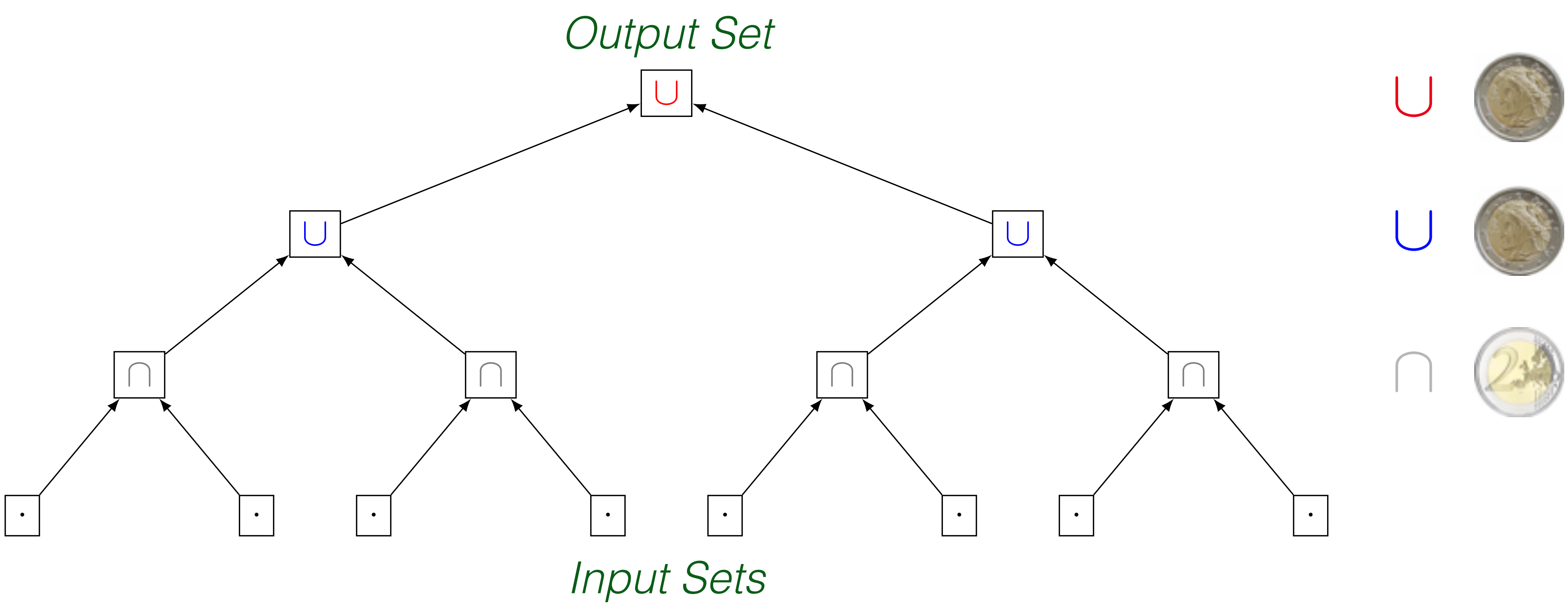
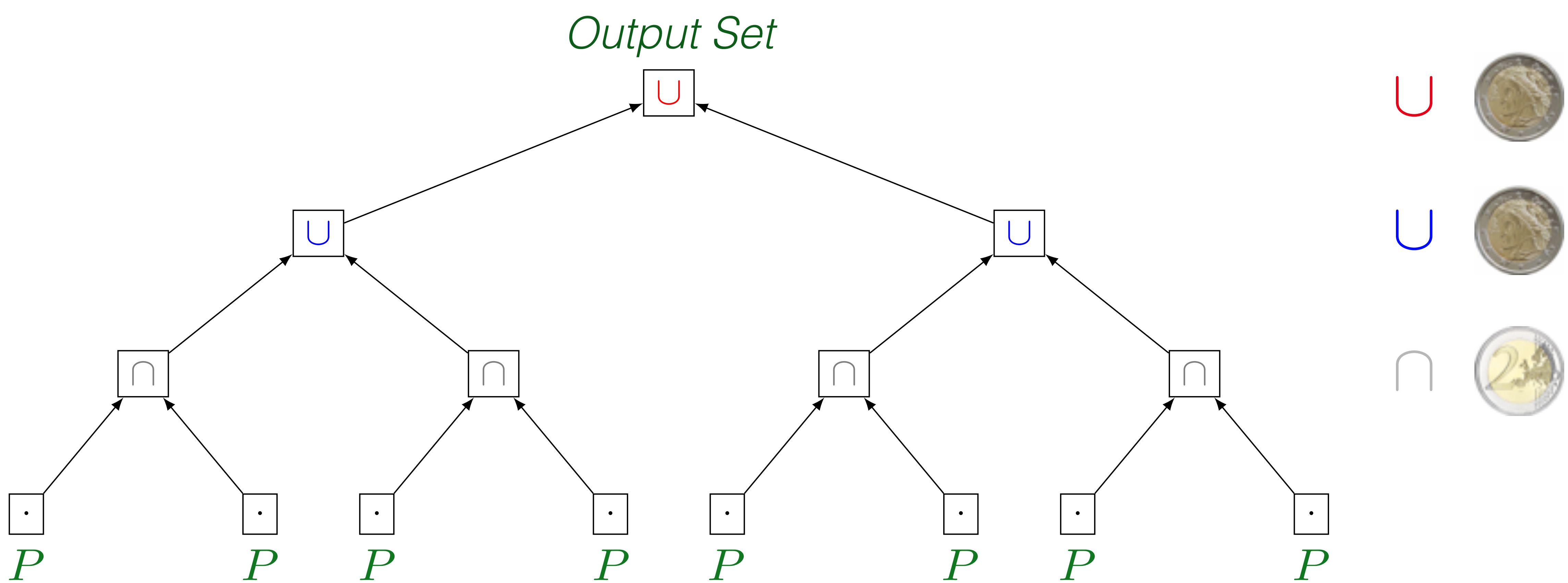
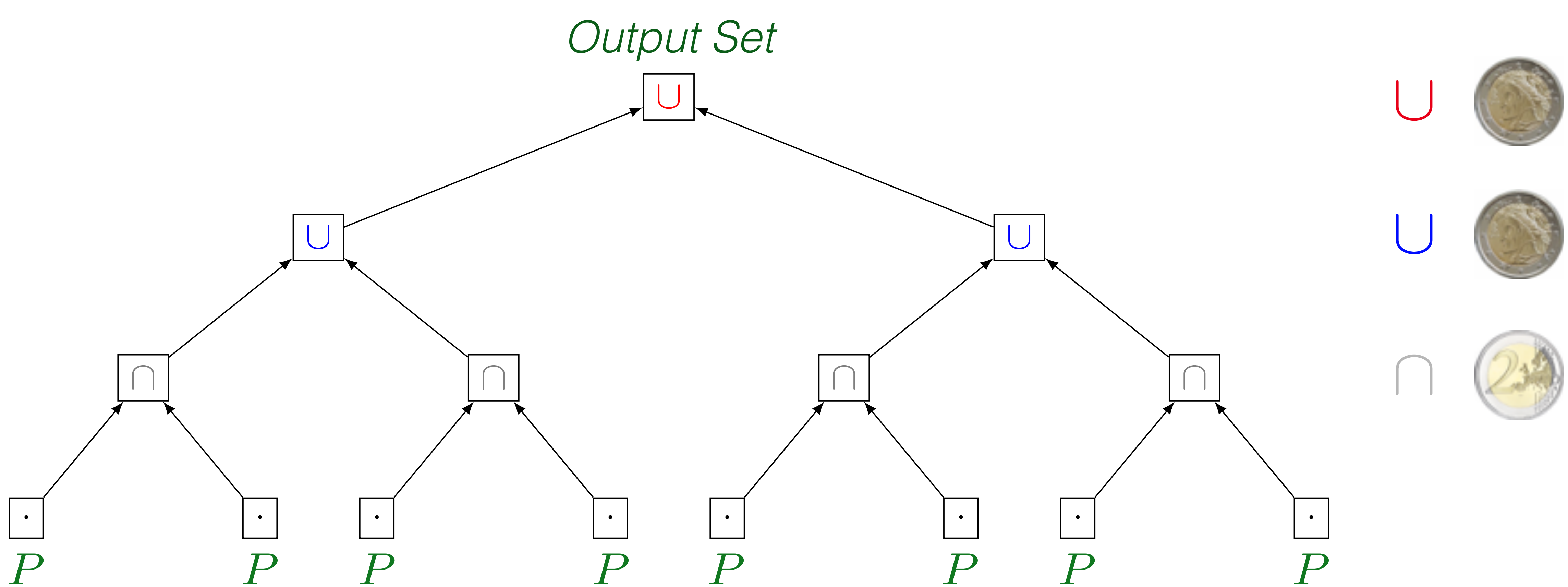
Striped Tree



Random System of ∪/∩ Striped Tree

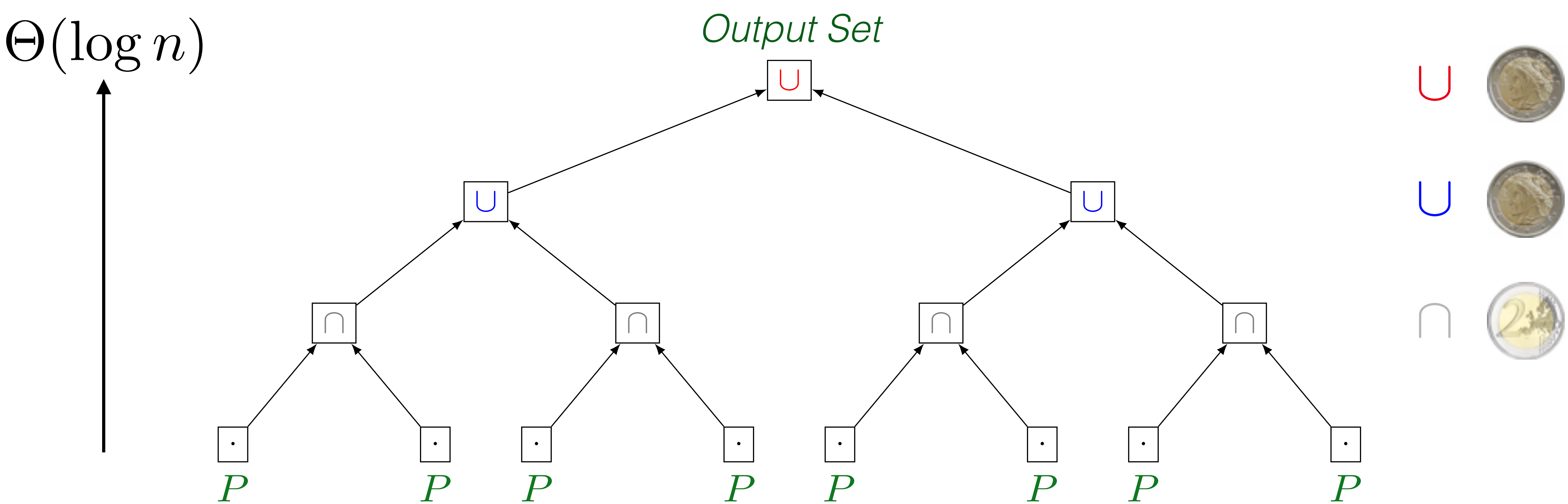


Random System of ∪/∩ Striped Tree



We prove (using the *random iterated systems'* framework) that choosing the levels' actions this way makes the tree act in a way "close to" a UAR threshold in *[0,1]*

Random System of $\cup/\cap$ Striped Tree



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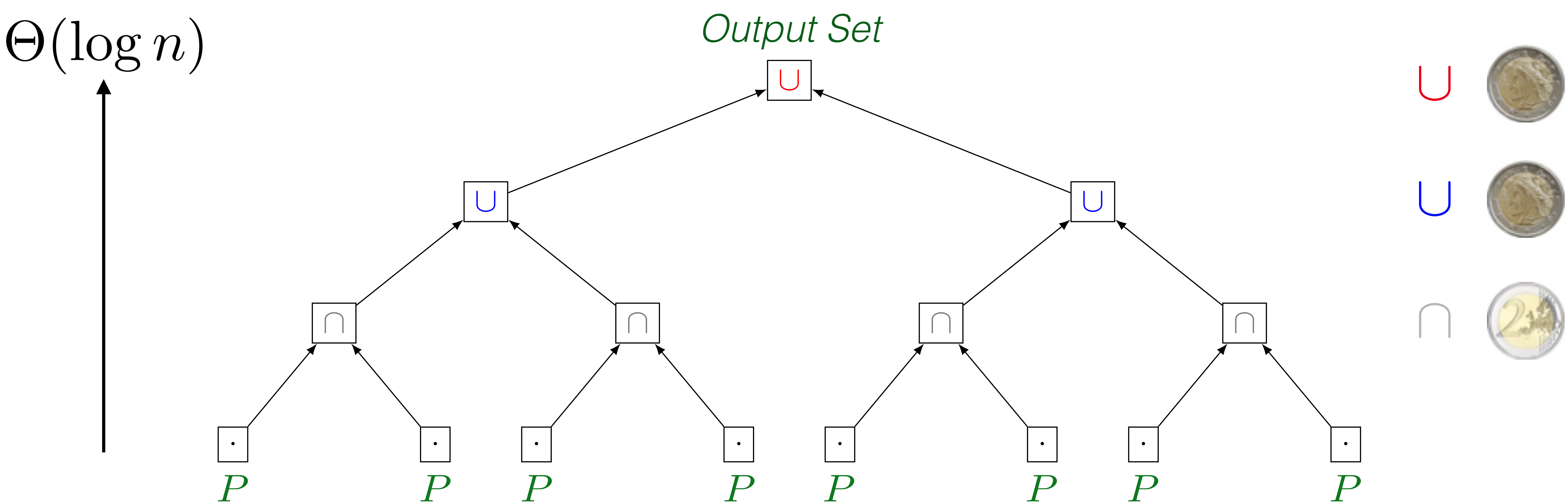
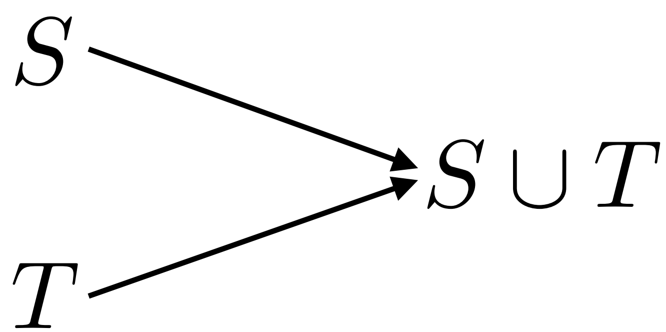
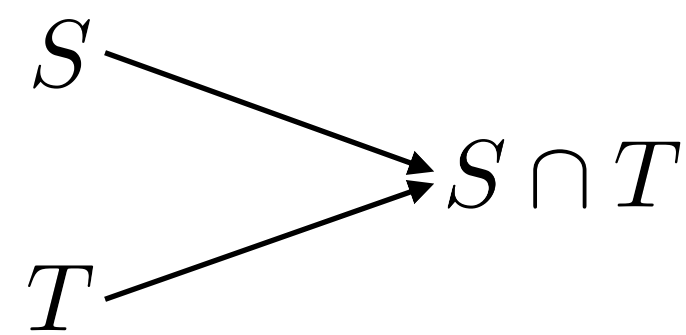
Random System of $\cup/\cap$ Striped Tree



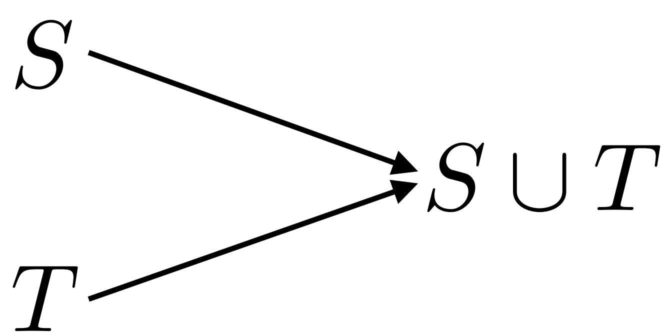
Thus, if we use the *same* random tree, once with input sets selected from P^+ and once from P^-, w.h.p. we will reach the same output set from both the **maximum** and the **minimum** sides

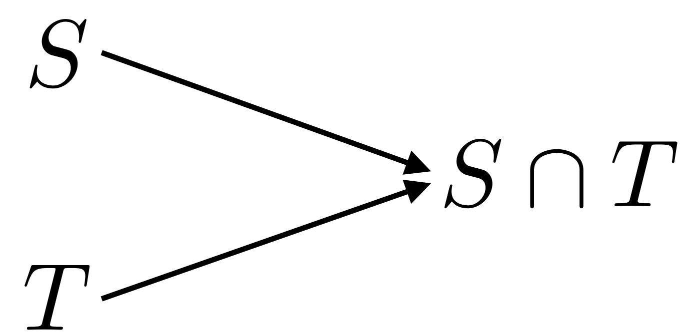
Rule Output(s)





Rule Output(s)





Approximately modular rules have *two* outputs

$$f(S) + f(T) = f(S \cup T) + f(S \cap T) \pm \epsilon$$

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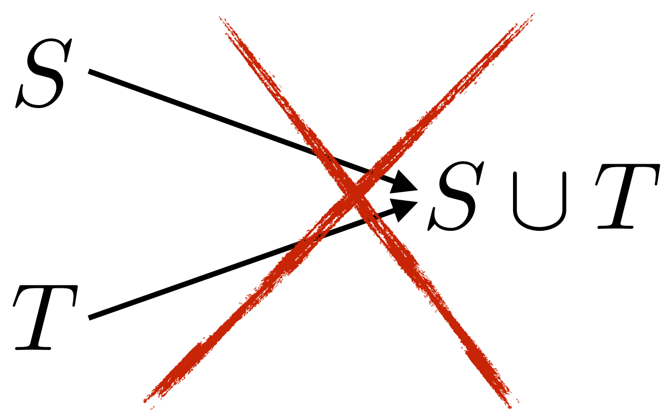

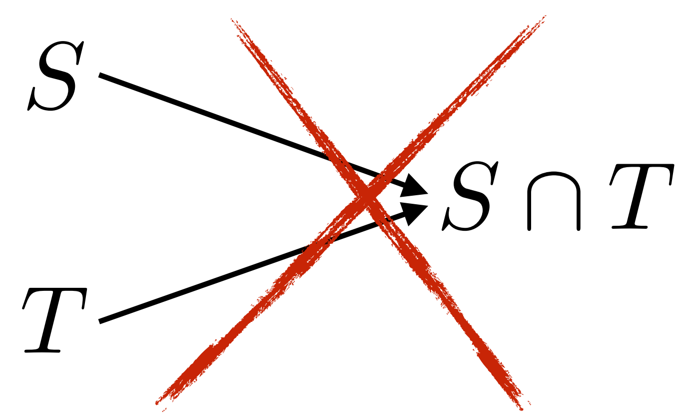

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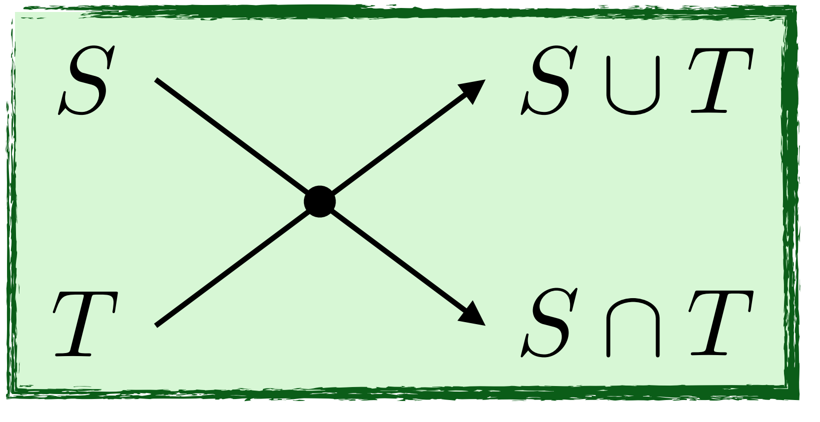
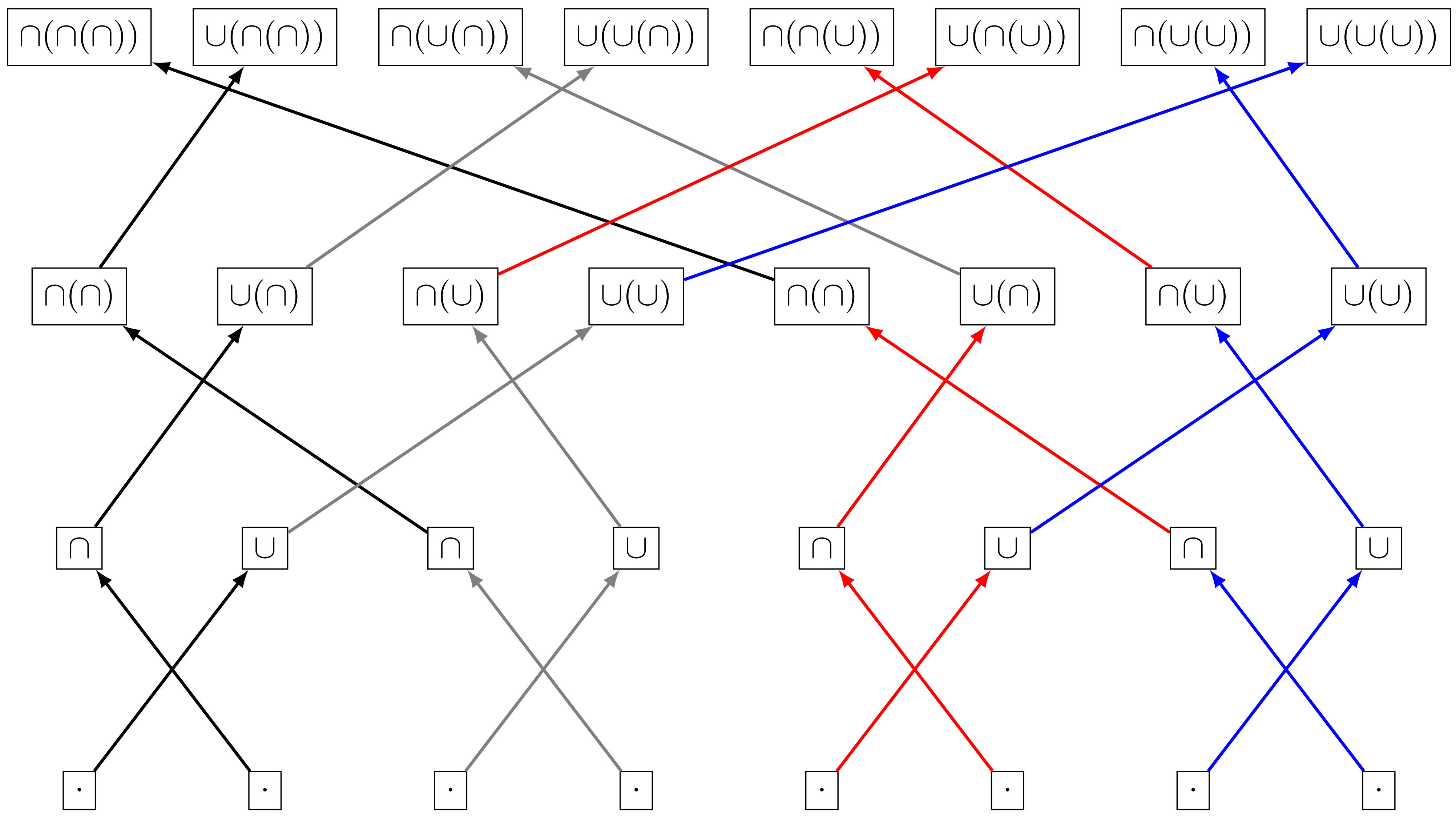


Striped Networks

We use *striped networks* to deal with the 2-outputs issue



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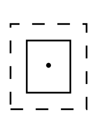

N_0

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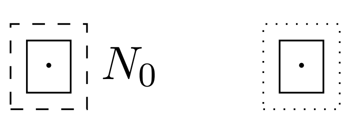

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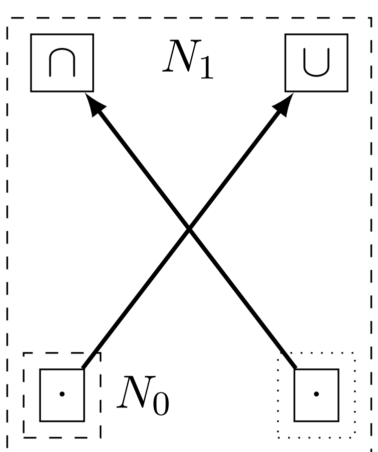

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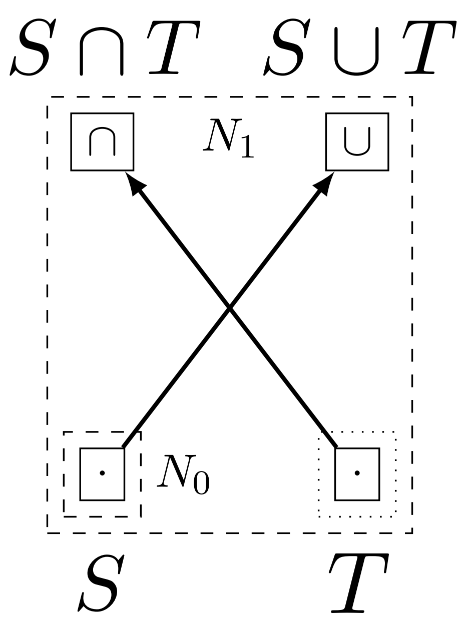

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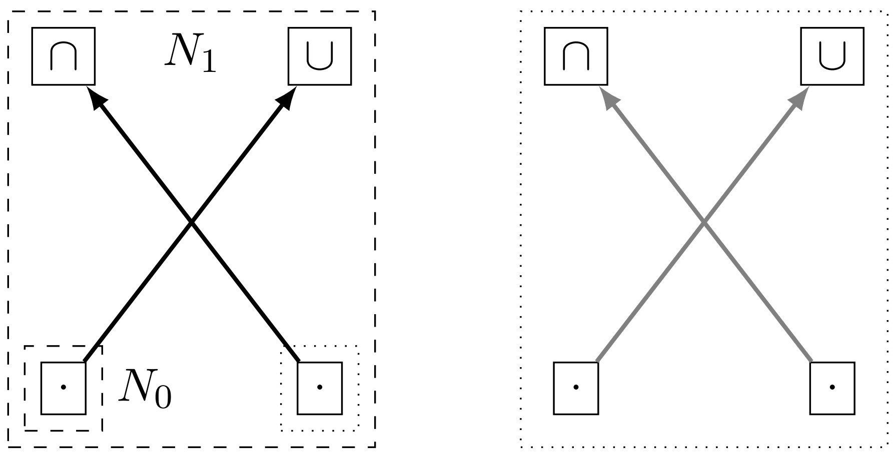

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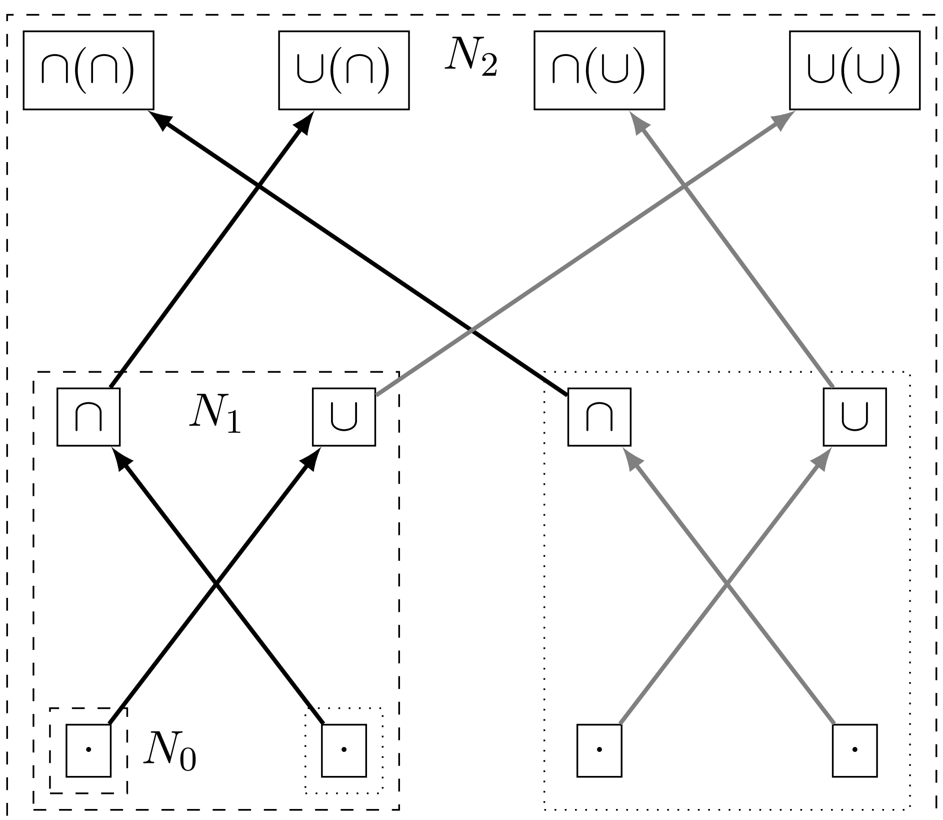

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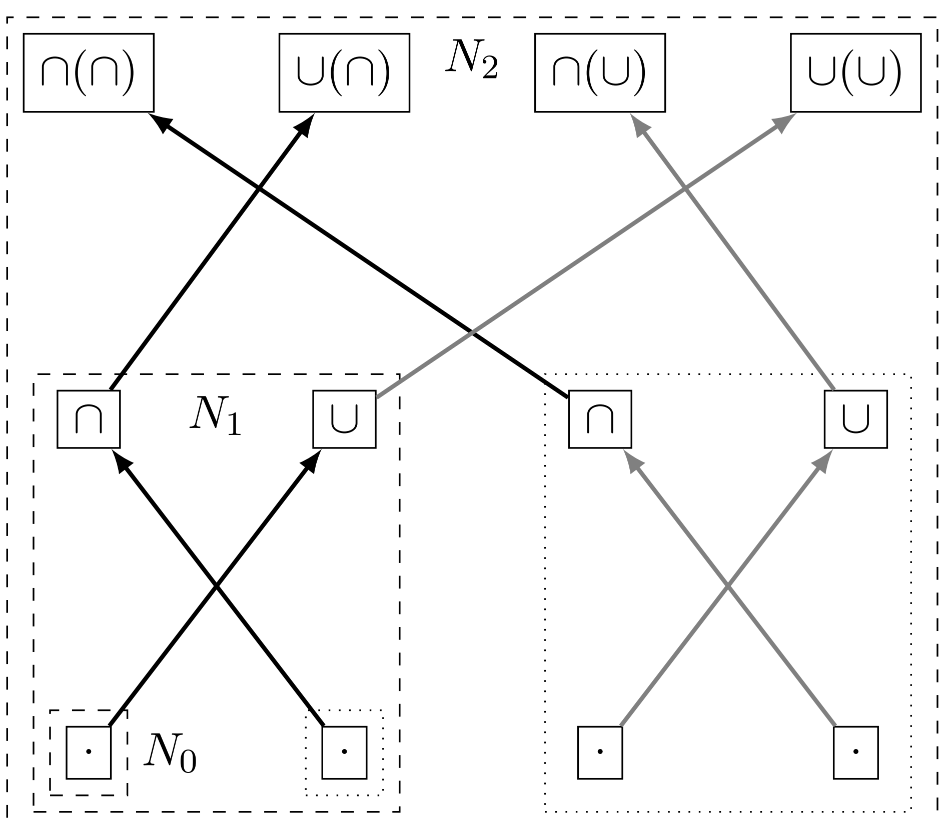

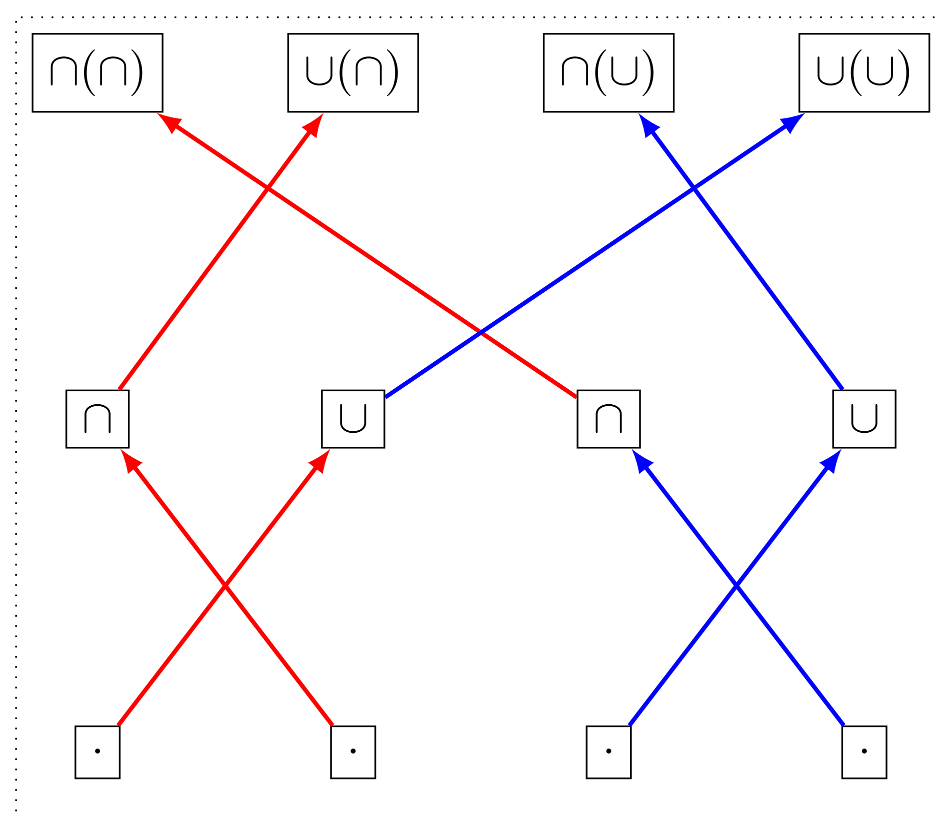

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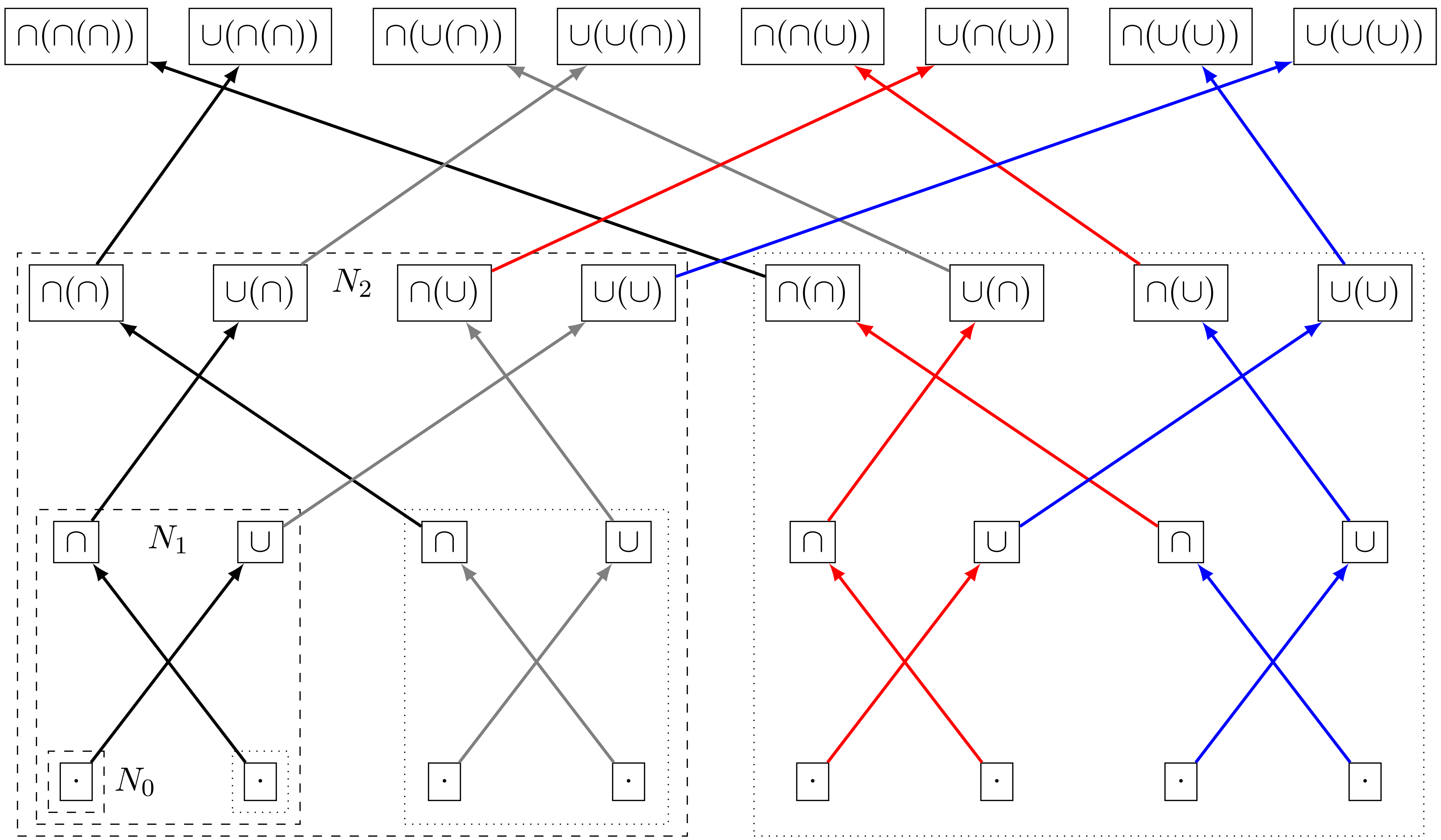
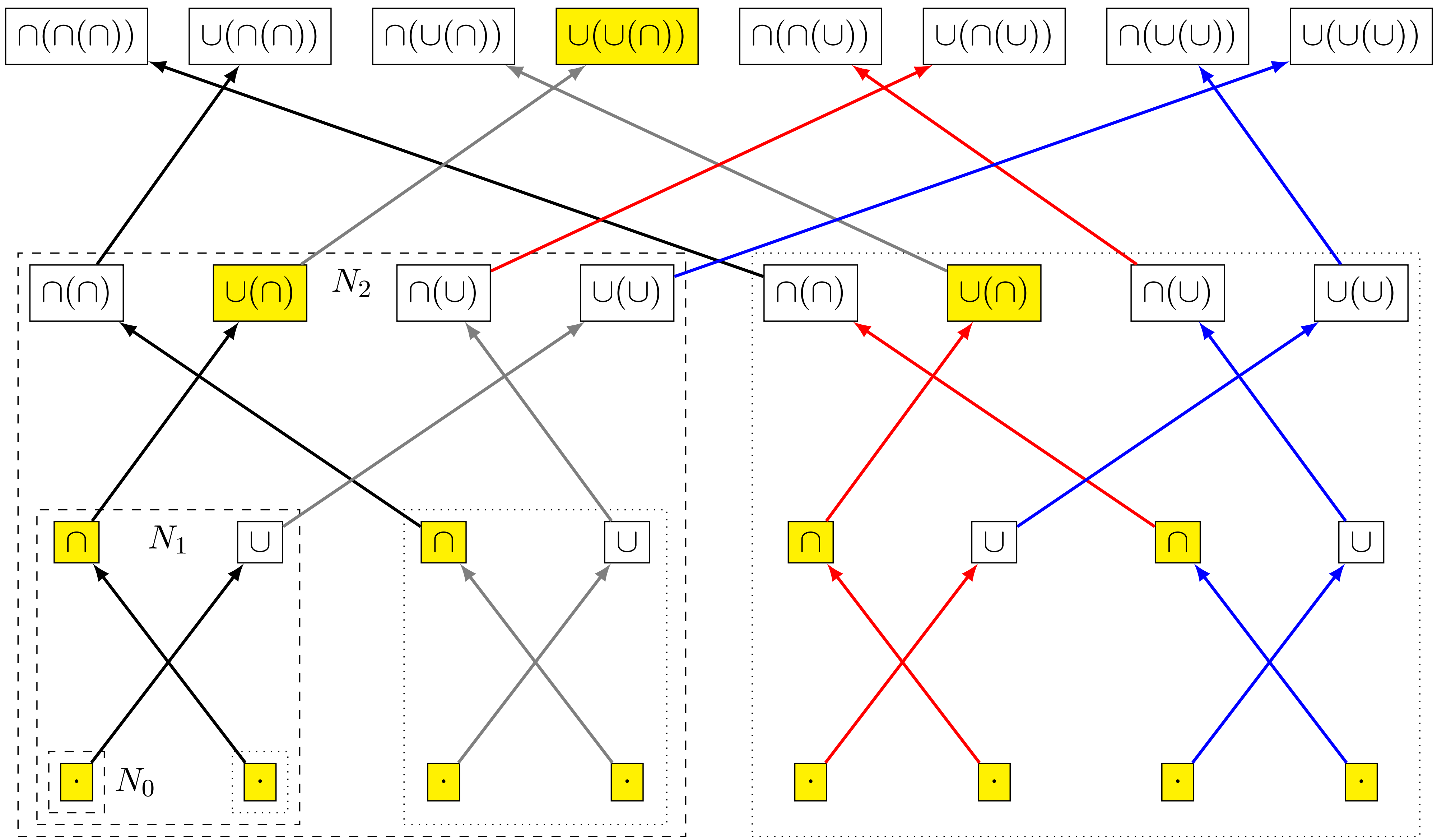


Striped Networks

Striped networks contain all the striped trees of a given height



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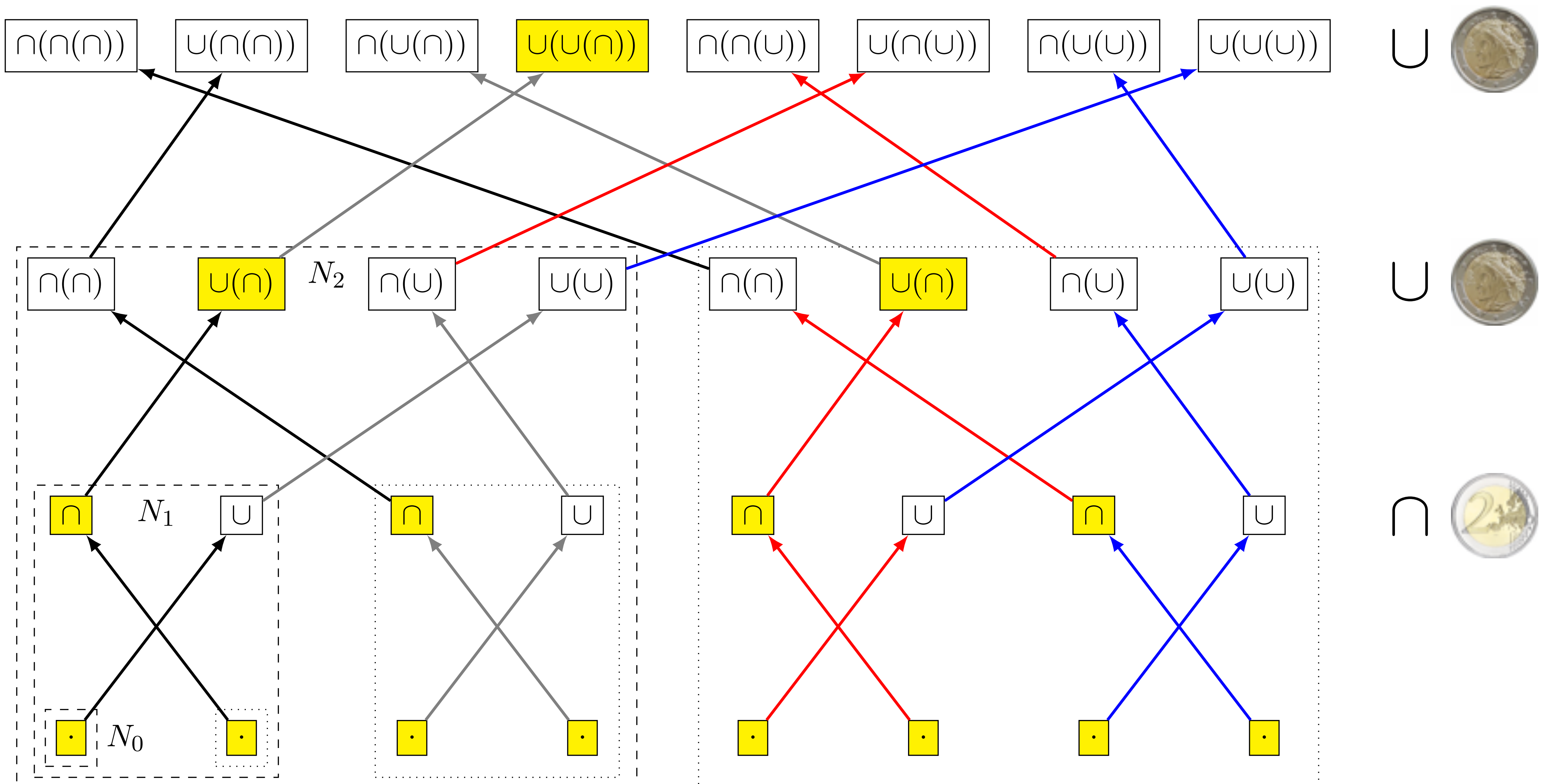

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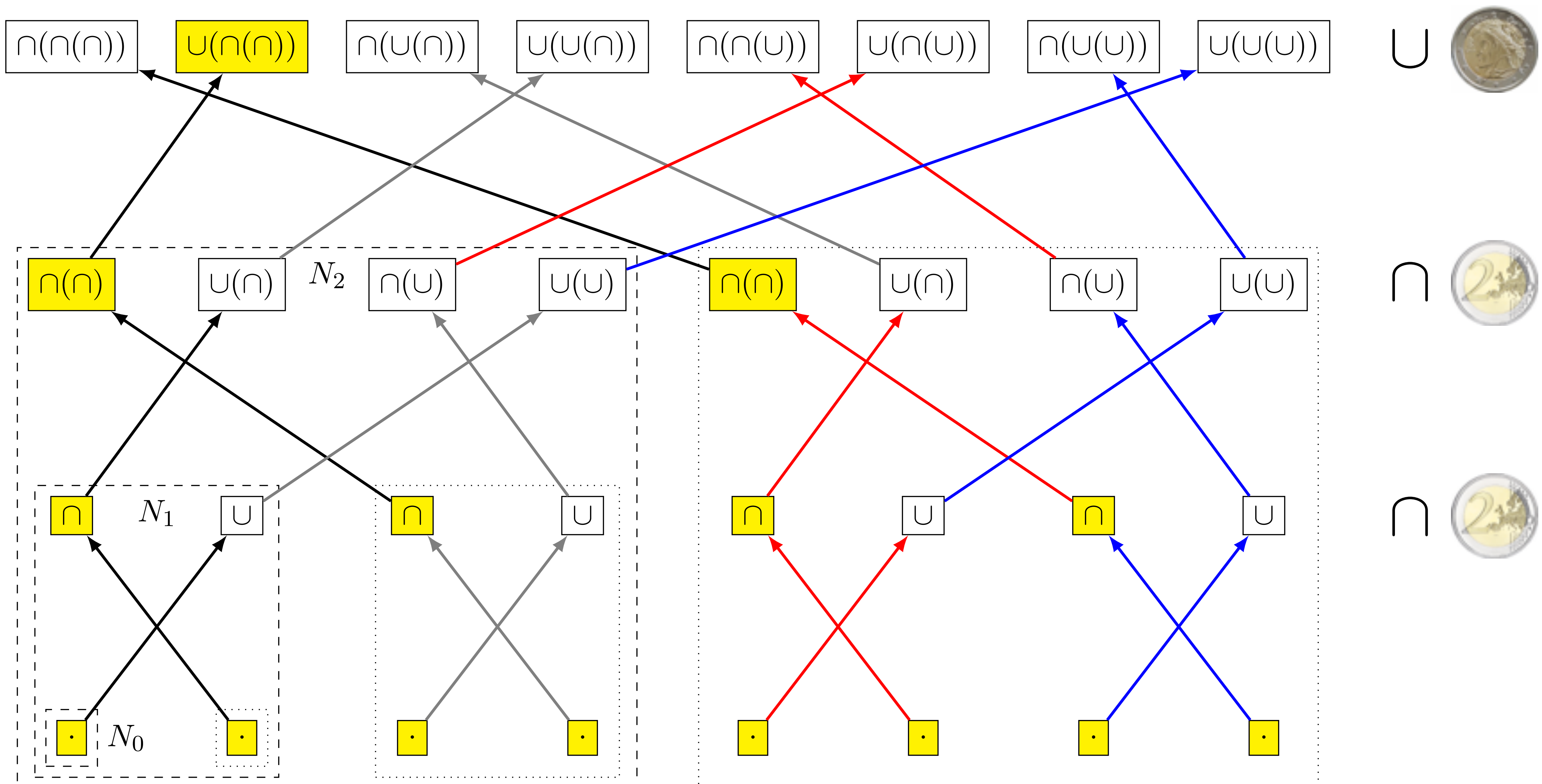

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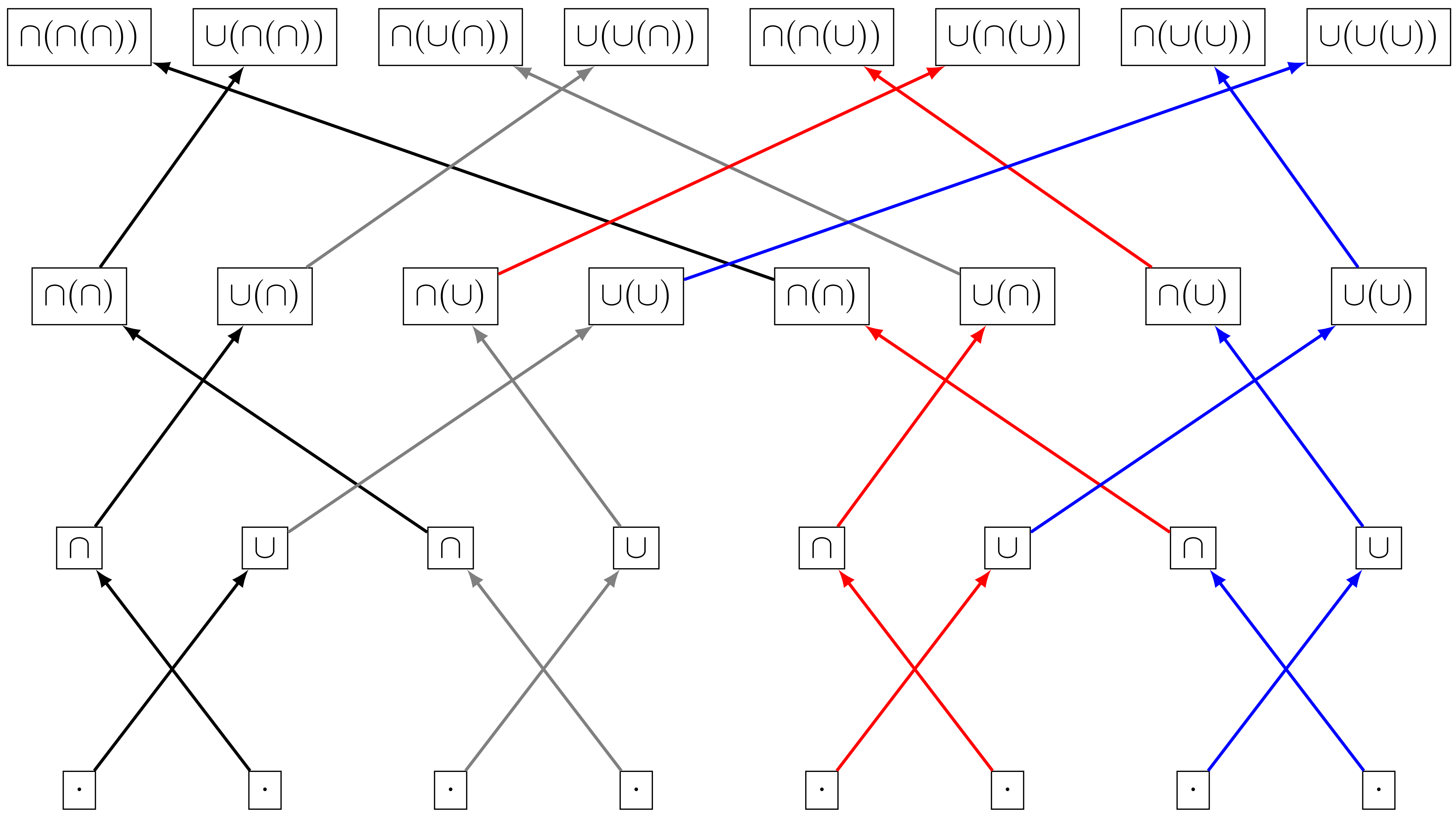
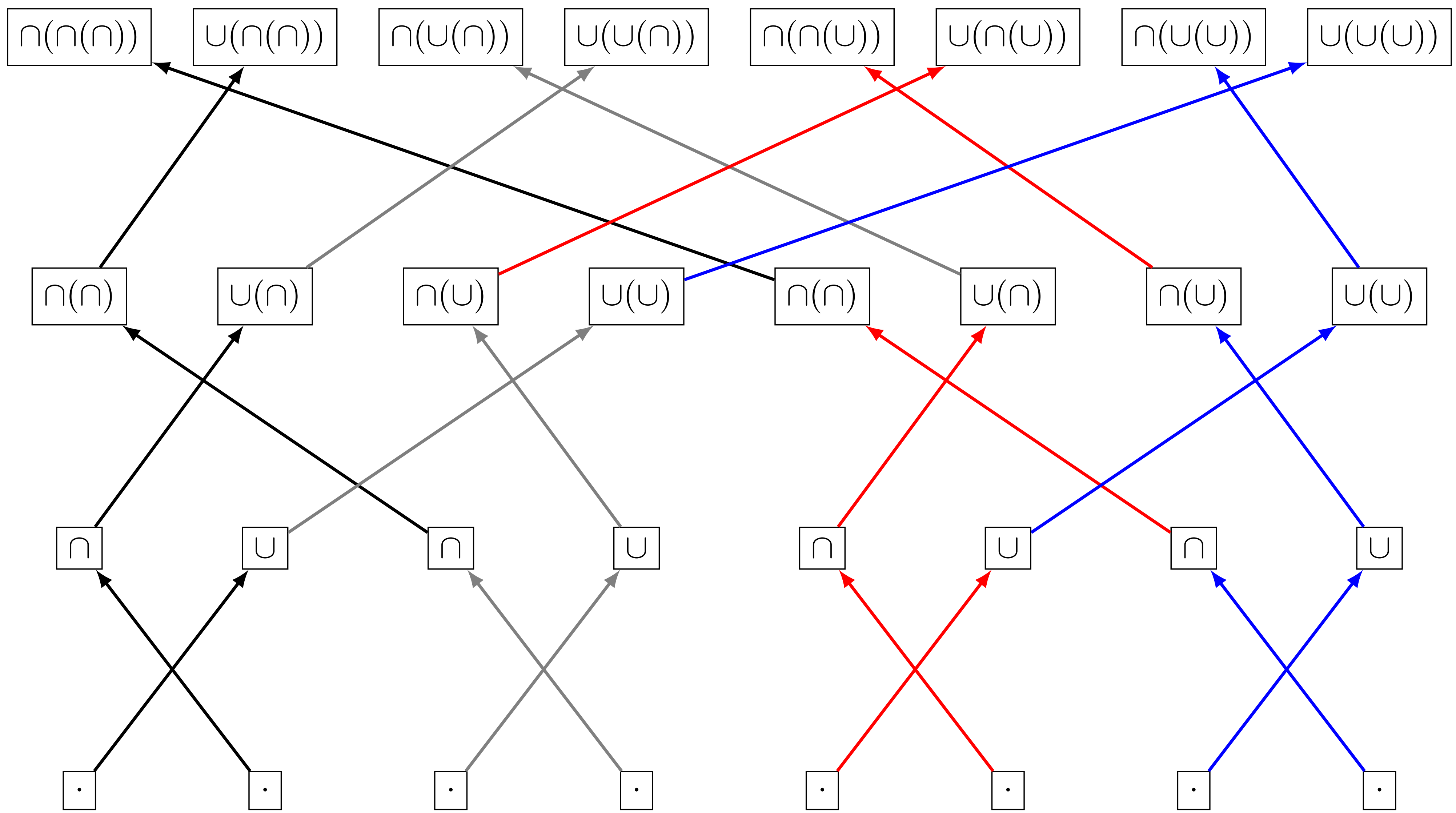
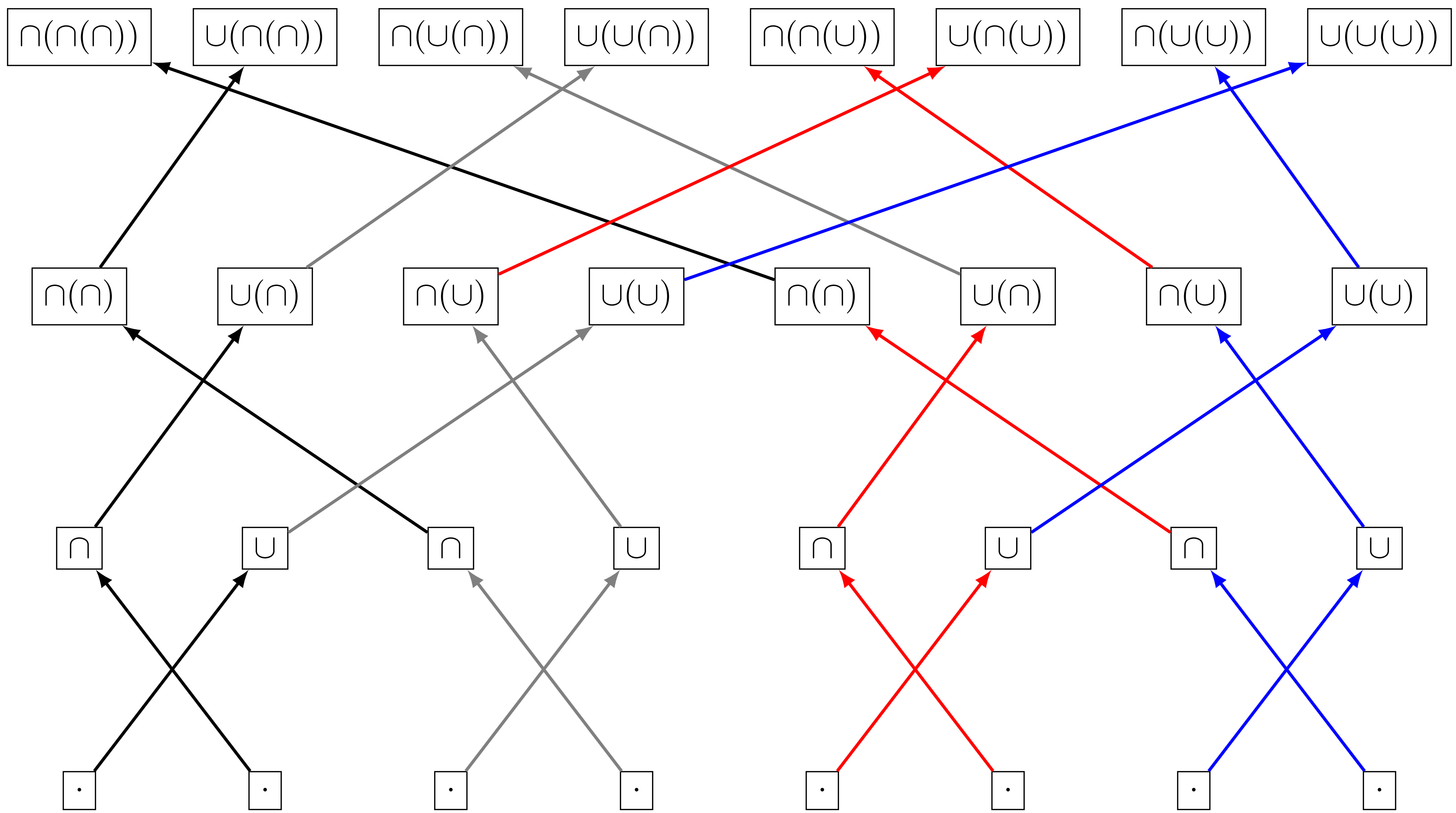


Striped Networks



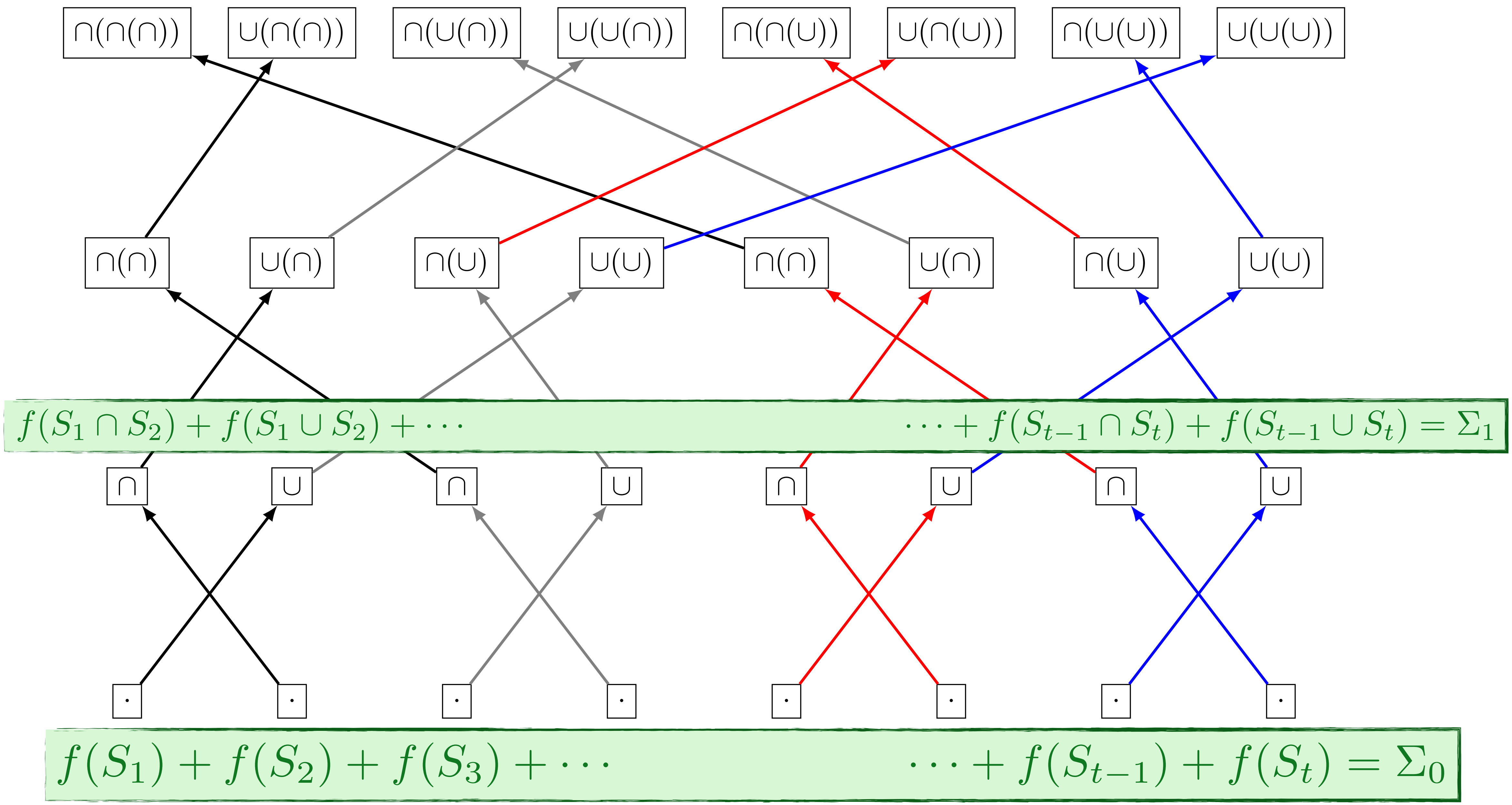
$f(S_1) + f(S_2) + f(S_3) + \cdots$ $\cdots + f(S_{t-1}) + f(S_t)$

Striped Networks

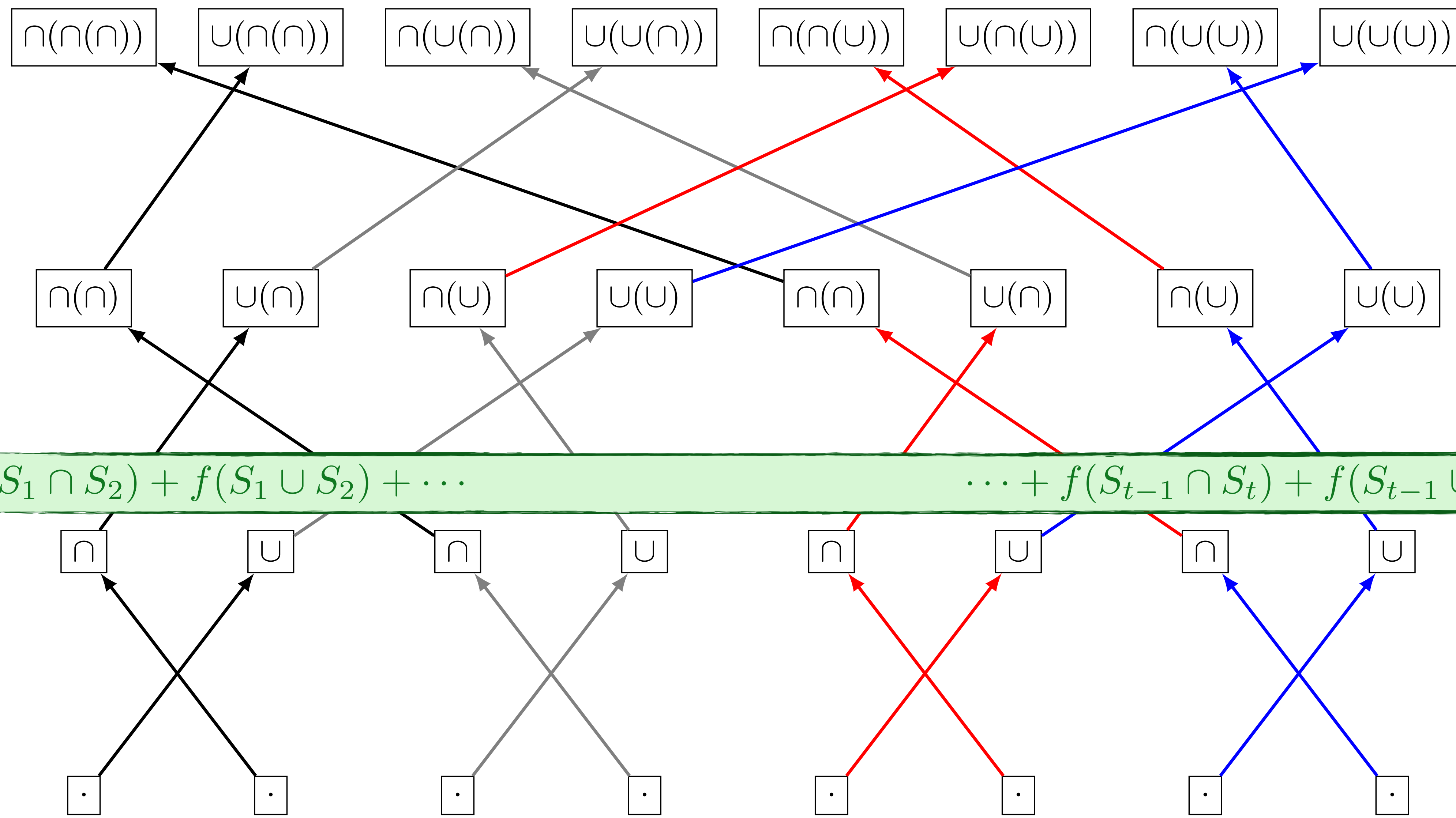


$f(S_1) + f(S_2) + f(S_3) + \cdots \qquad \cdots + f(S_{t-1}) + f(S_t) = \Sigma_0$

Striped Networks



Striped Networks

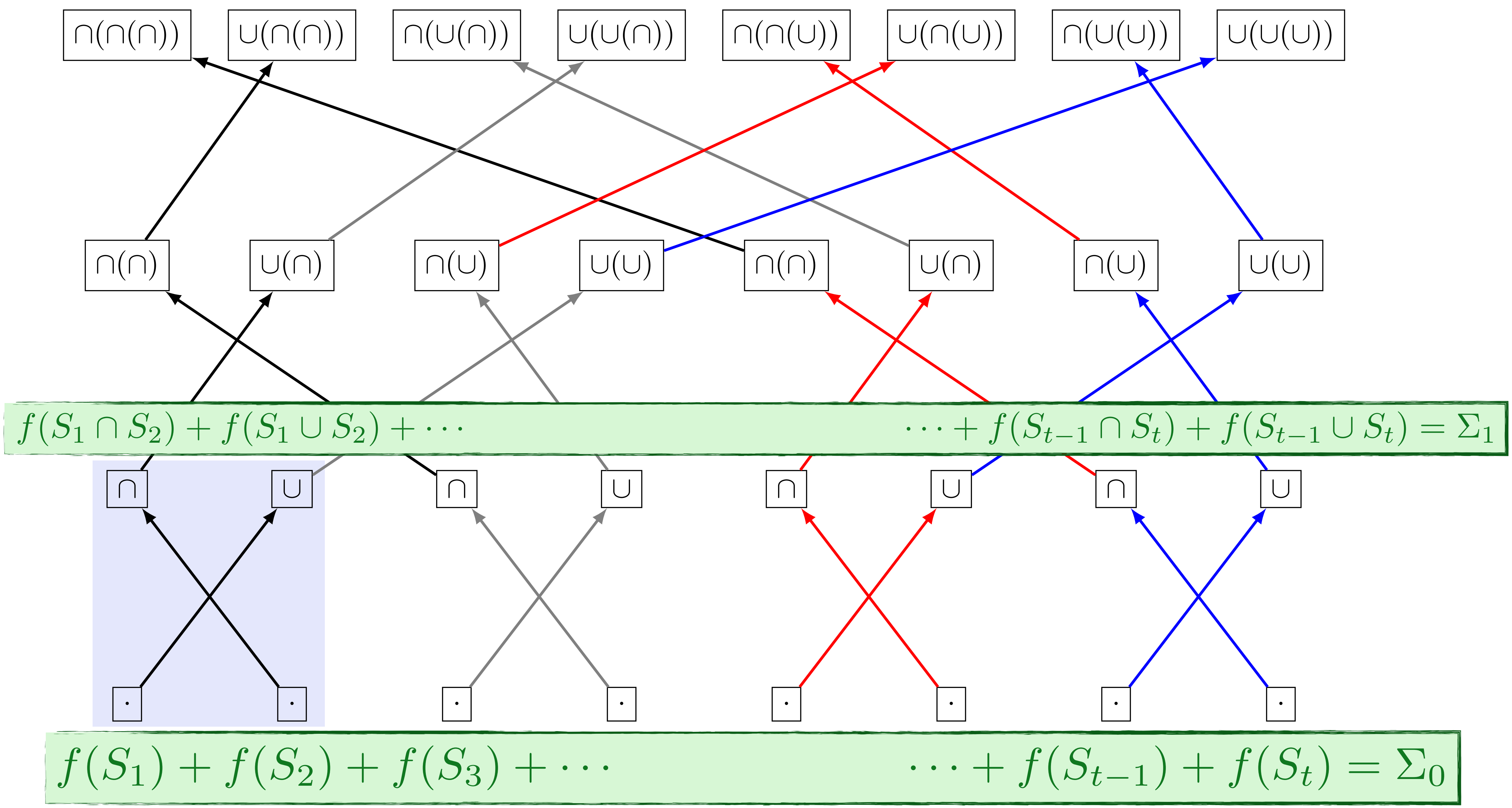


$$f(S_1 \cap S_2) + f(S_1 \cup S_2) + \cdots \quad \cdots + f(S_{t-1} \cap S_t) + f(S_{t-1} \cup S_t) = \Sigma_1$$

$$f(S_1) + f(S_2) + f(S_3) + \cdots \quad \cdots + f(S_{t-1}) + f(S_t) = \Sigma_0$$

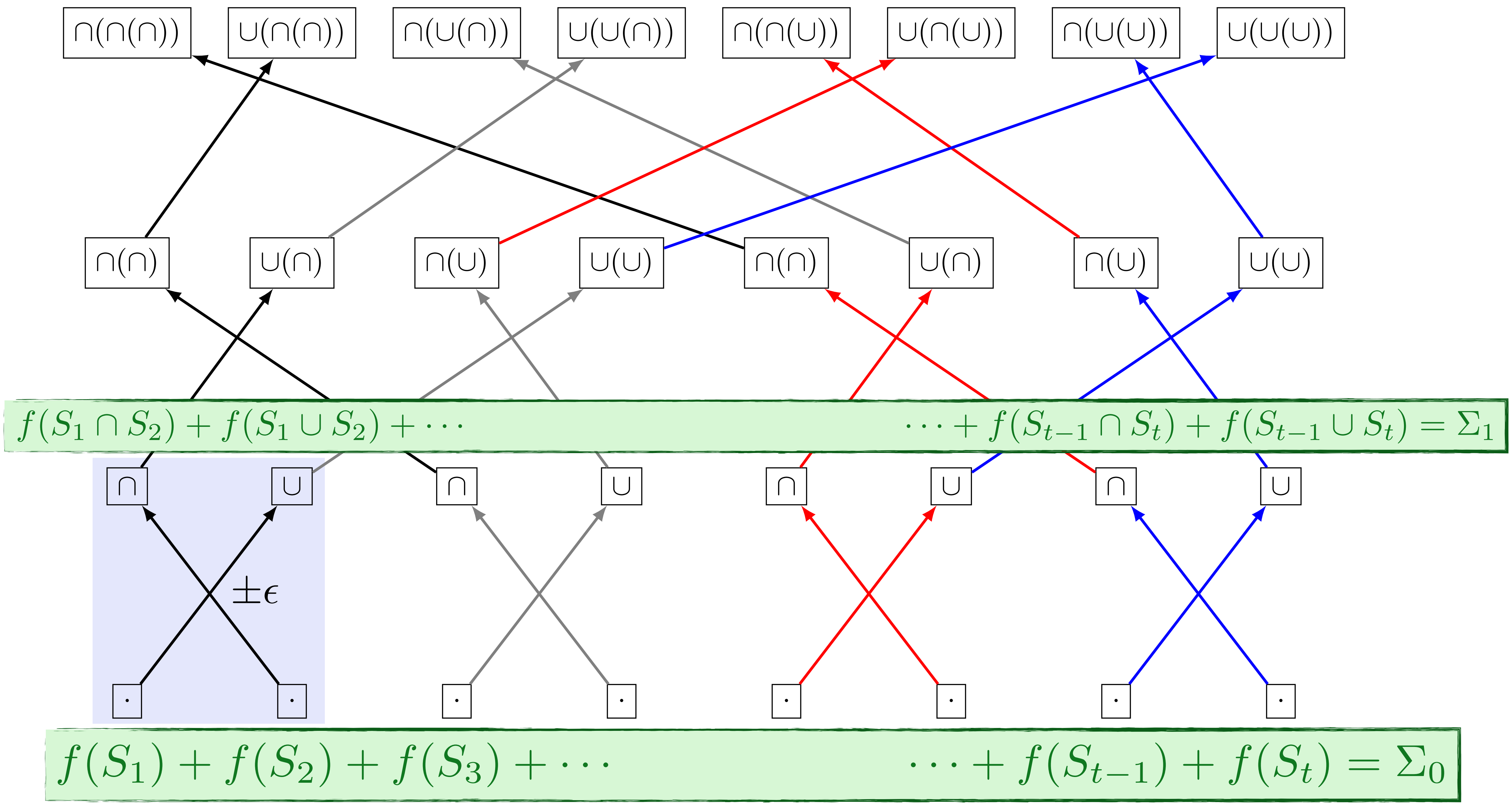
$$f(S \cup T) + f(S \cap T) - \epsilon \leq f(S) + f(T) \leq f(S \cup T) + f(S \cap T) + \epsilon$$

Striped Networks



$$f(S \cup T) + f(S \cap T) - \epsilon \leq f(S) + f(T) \leq f(S \cup T) + f(S \cap T) + \epsilon$$

Striped Networks



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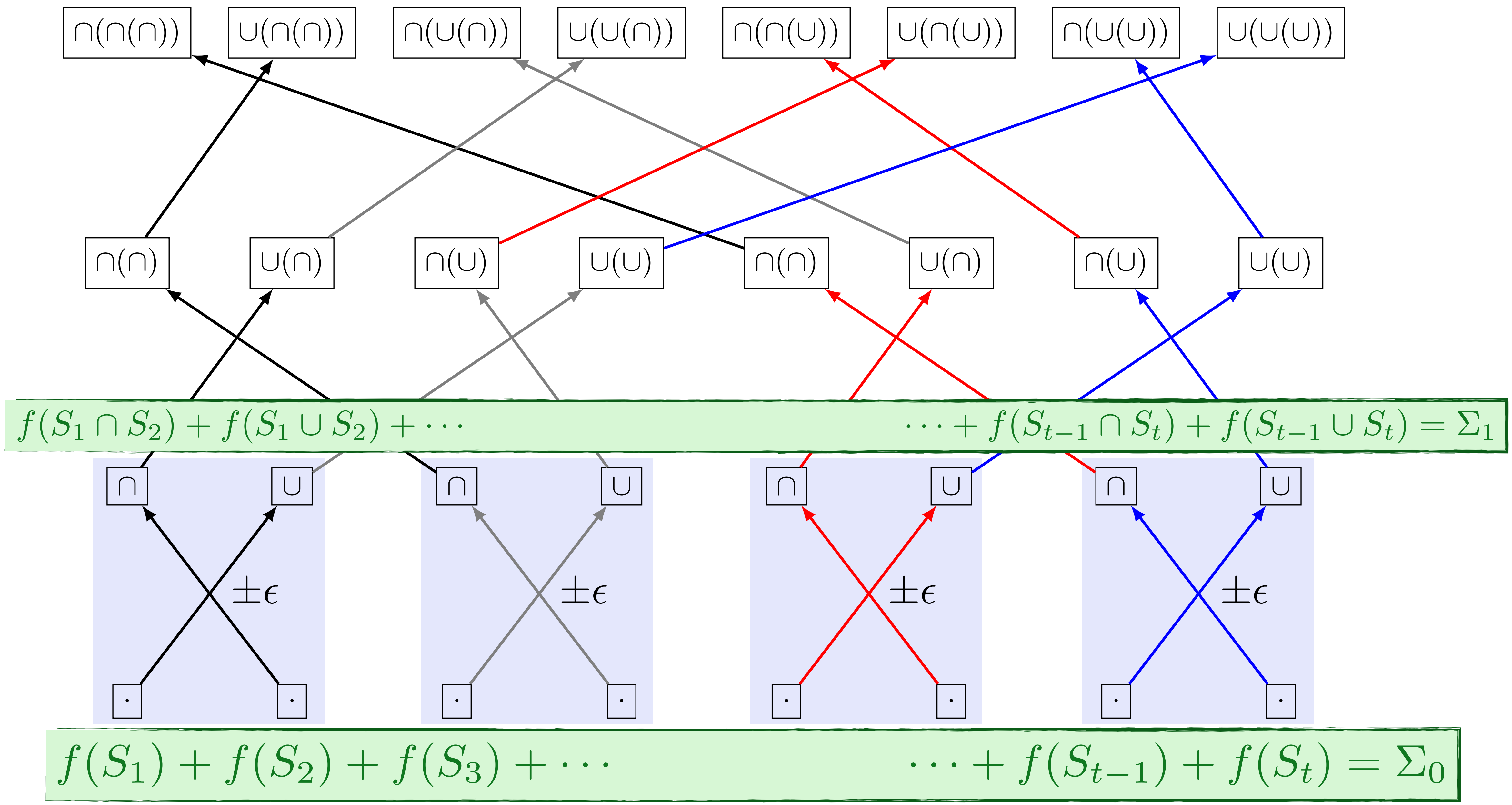
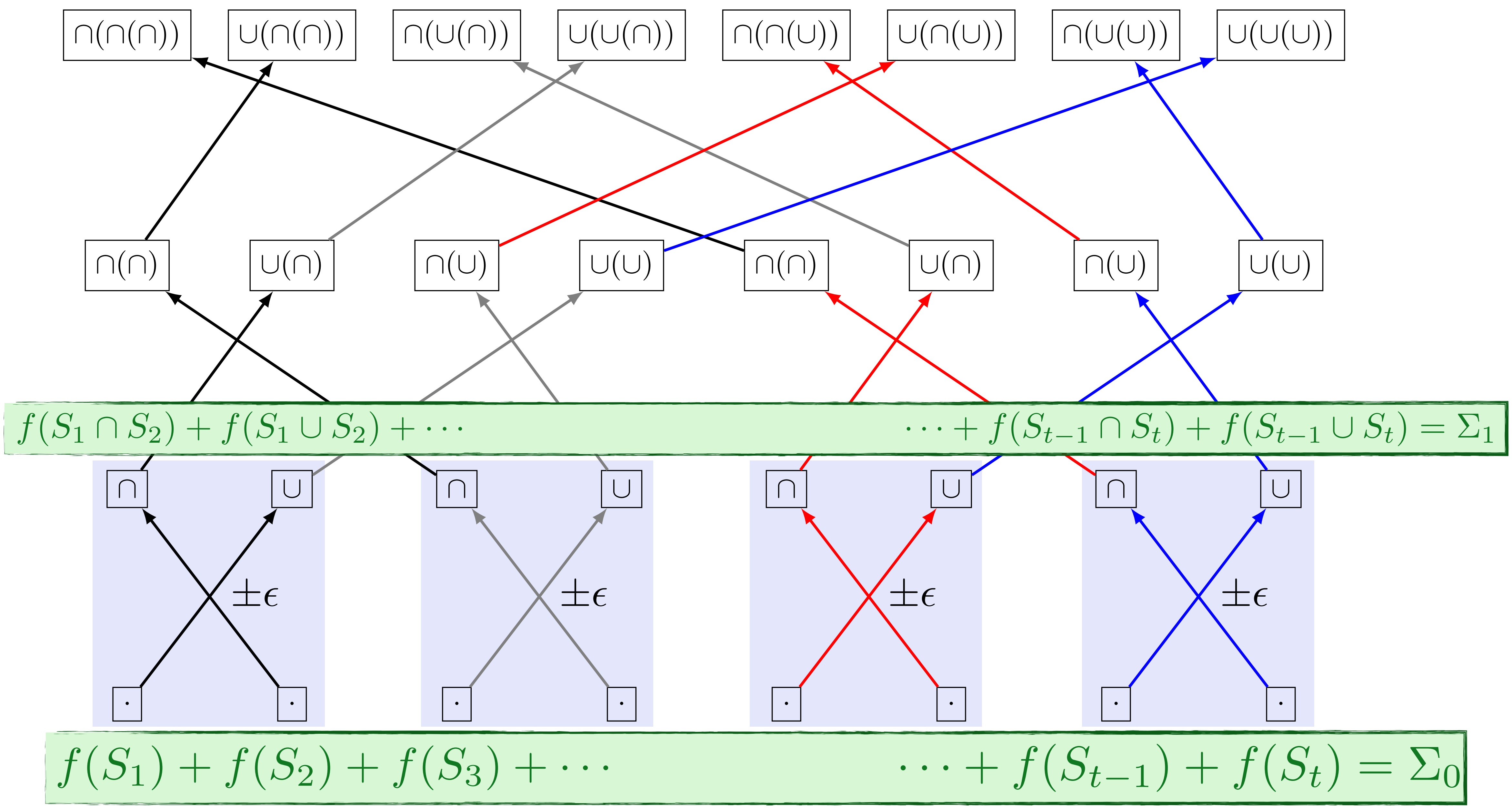
Striped Networks

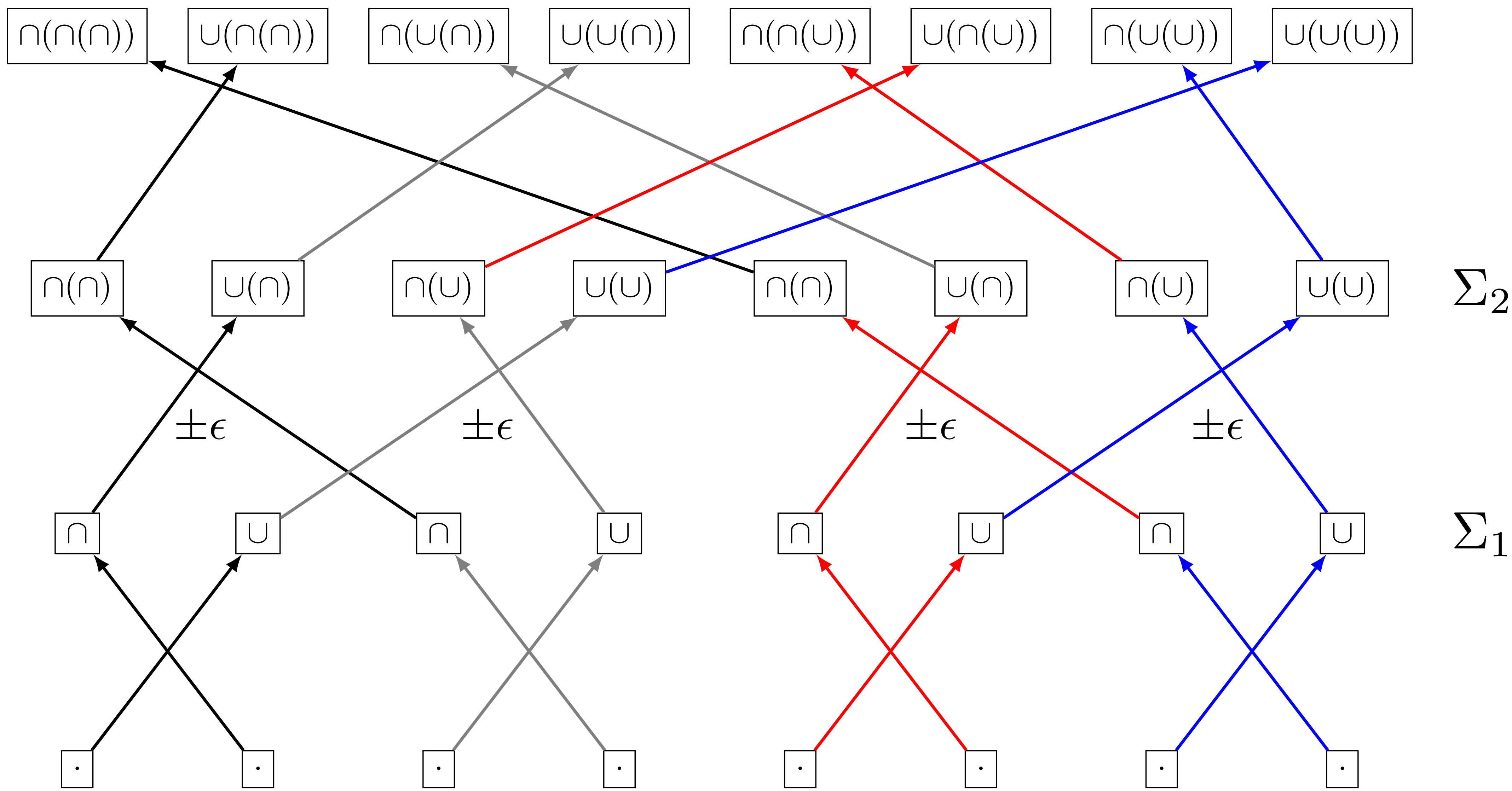


Striped Networks



$$\Sigma_0 = \Sigma_1 \pm \epsilon \cdot {}^{t}\!/_{2}$$

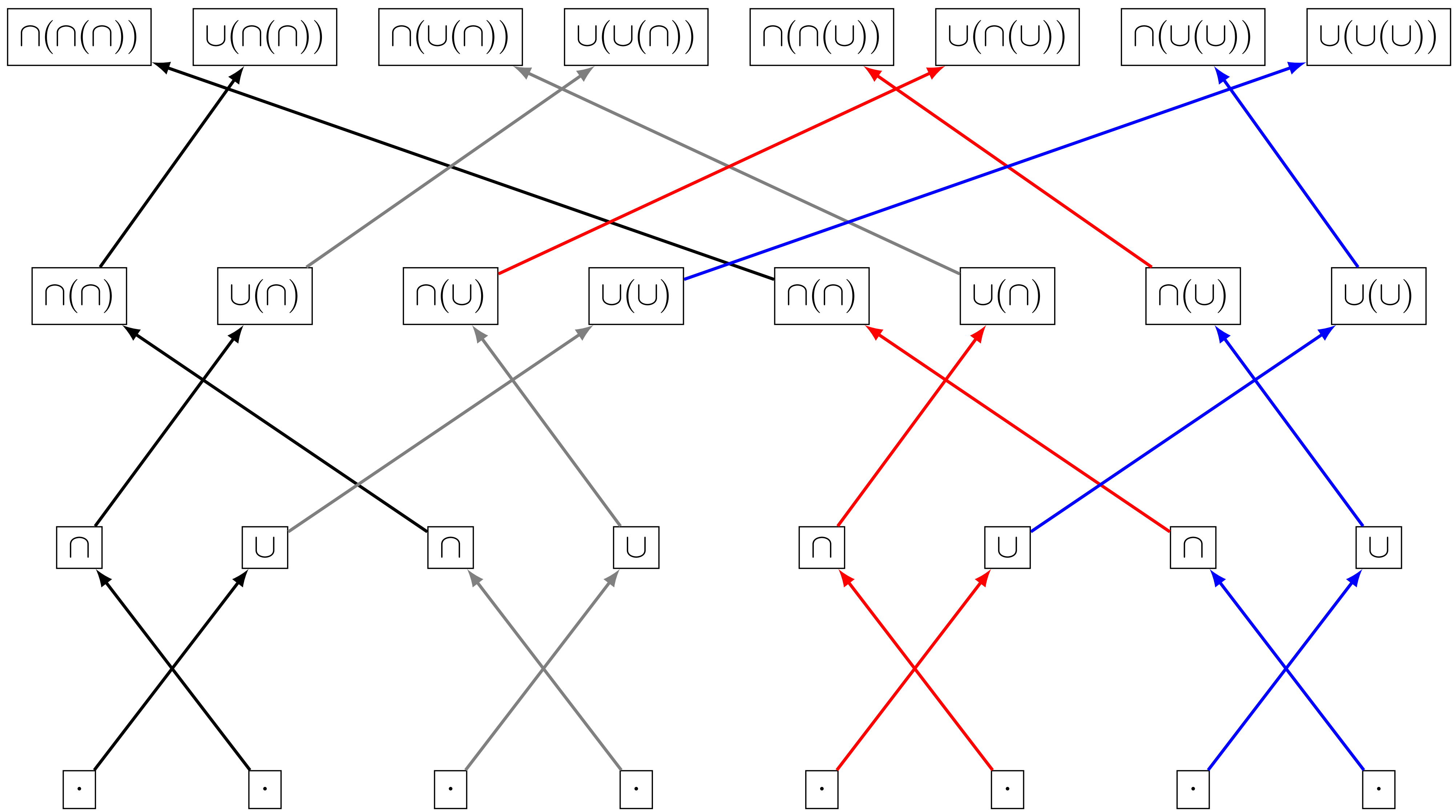
Striped Networks



$\Sigma_0 = \Sigma_1 \pm \epsilon \cdot {}^{t}\!/_{2}$ $\qquad$ $\Sigma_1 = \Sigma_2 \pm \epsilon \cdot {}^{t}\!/_{2}$

Striped Networks

Striped networks "lose" an additive $\epsilon/2$ average term per level

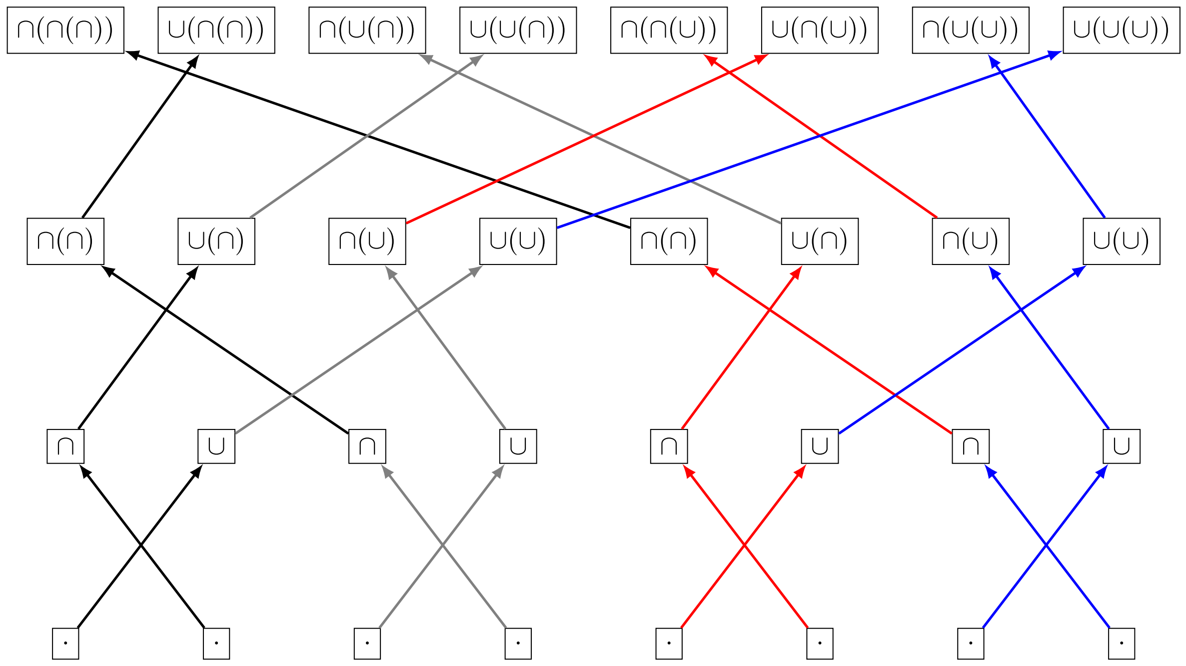


$\Sigma_0 = \Sigma_1 \pm \epsilon \cdot t/2$ $\Sigma_1 = \Sigma_2 \pm \epsilon \cdot t/2$ $\cdots$ $\Sigma_{h-1} = \Sigma_h \pm \epsilon \cdot t/2$

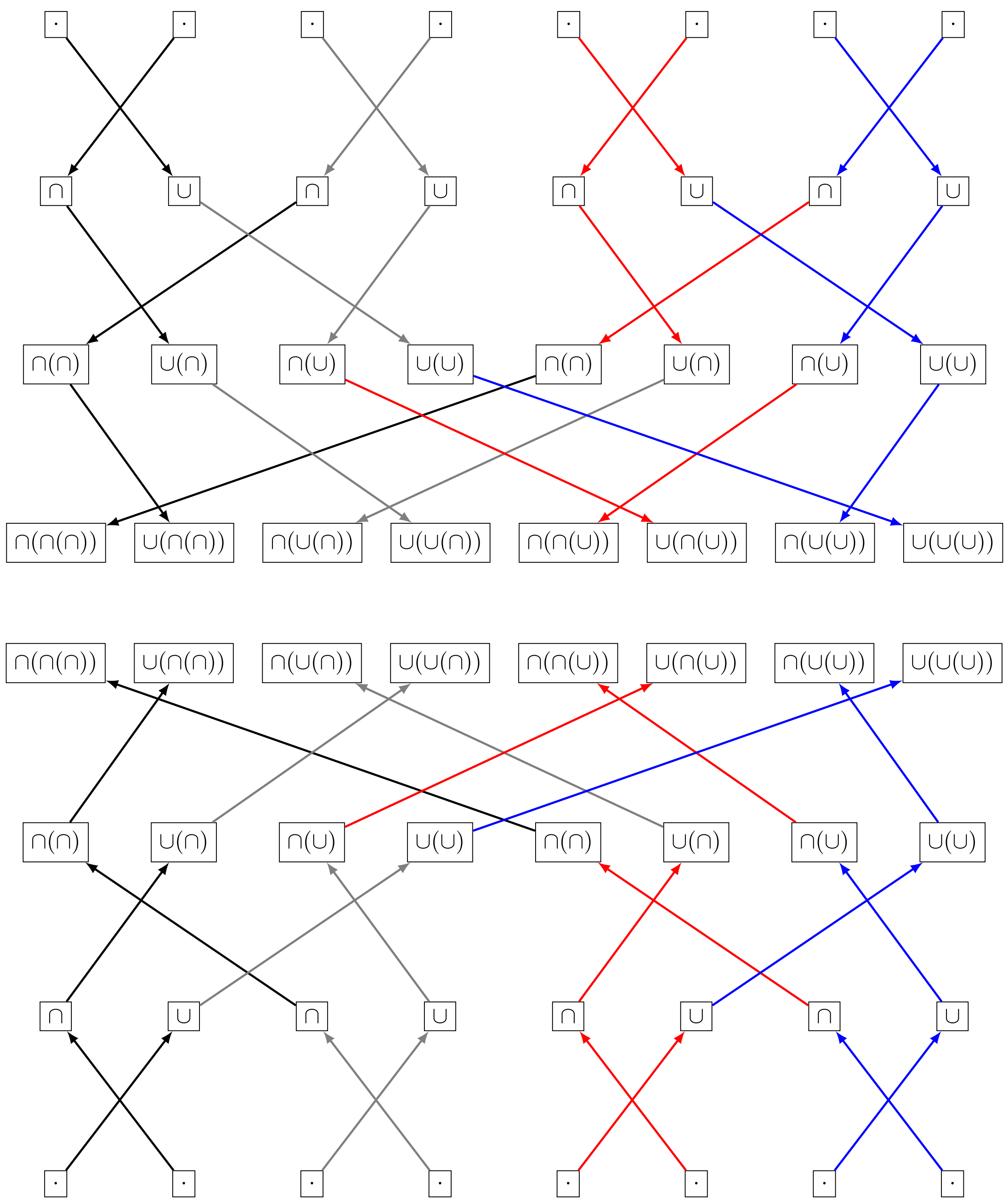
Bounding M

We use striped networks to bound the maximum value M of the ϵ-approximately modular functions F that are best approximated by z

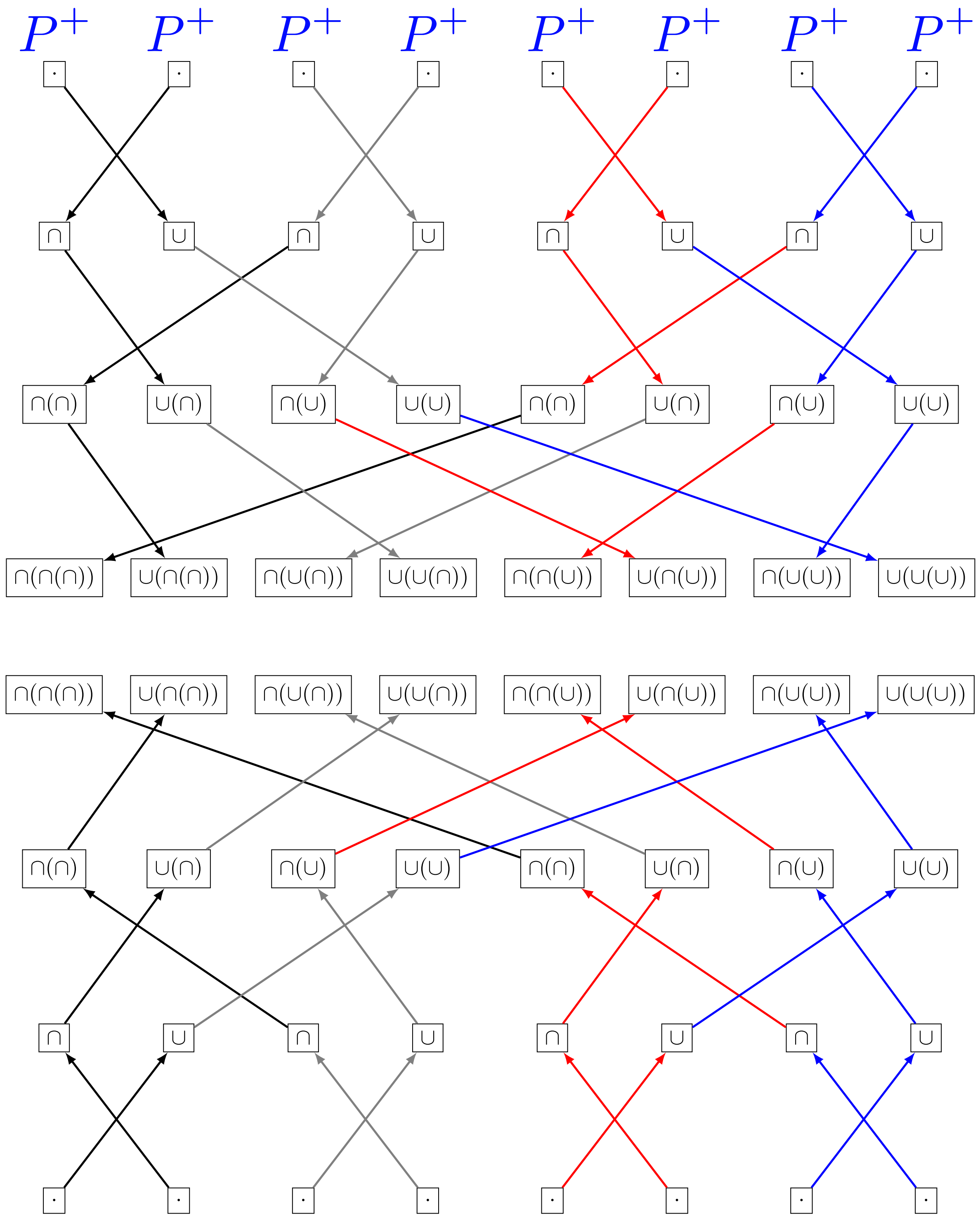
Bounding M



Bounding M



Bounding M



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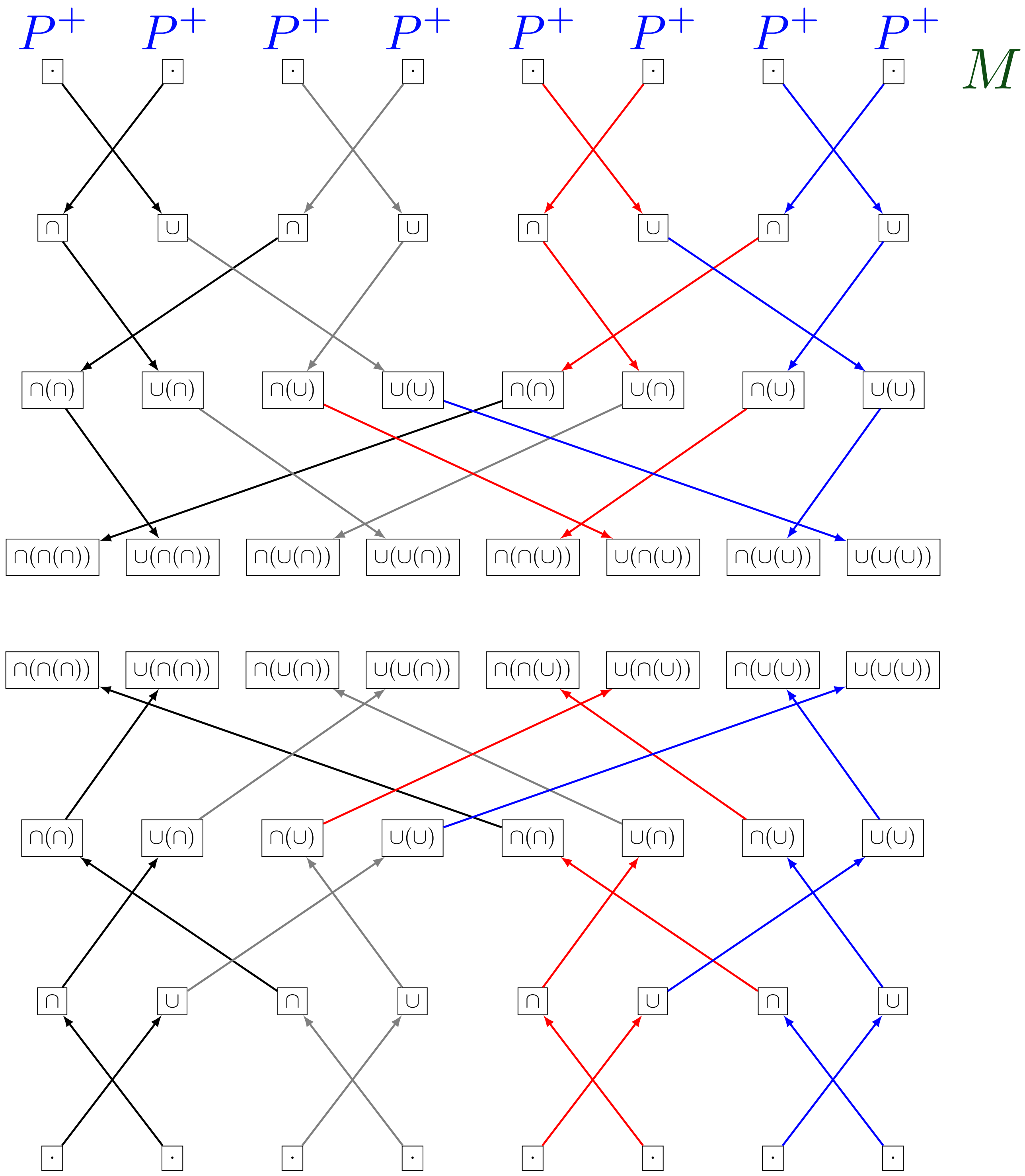

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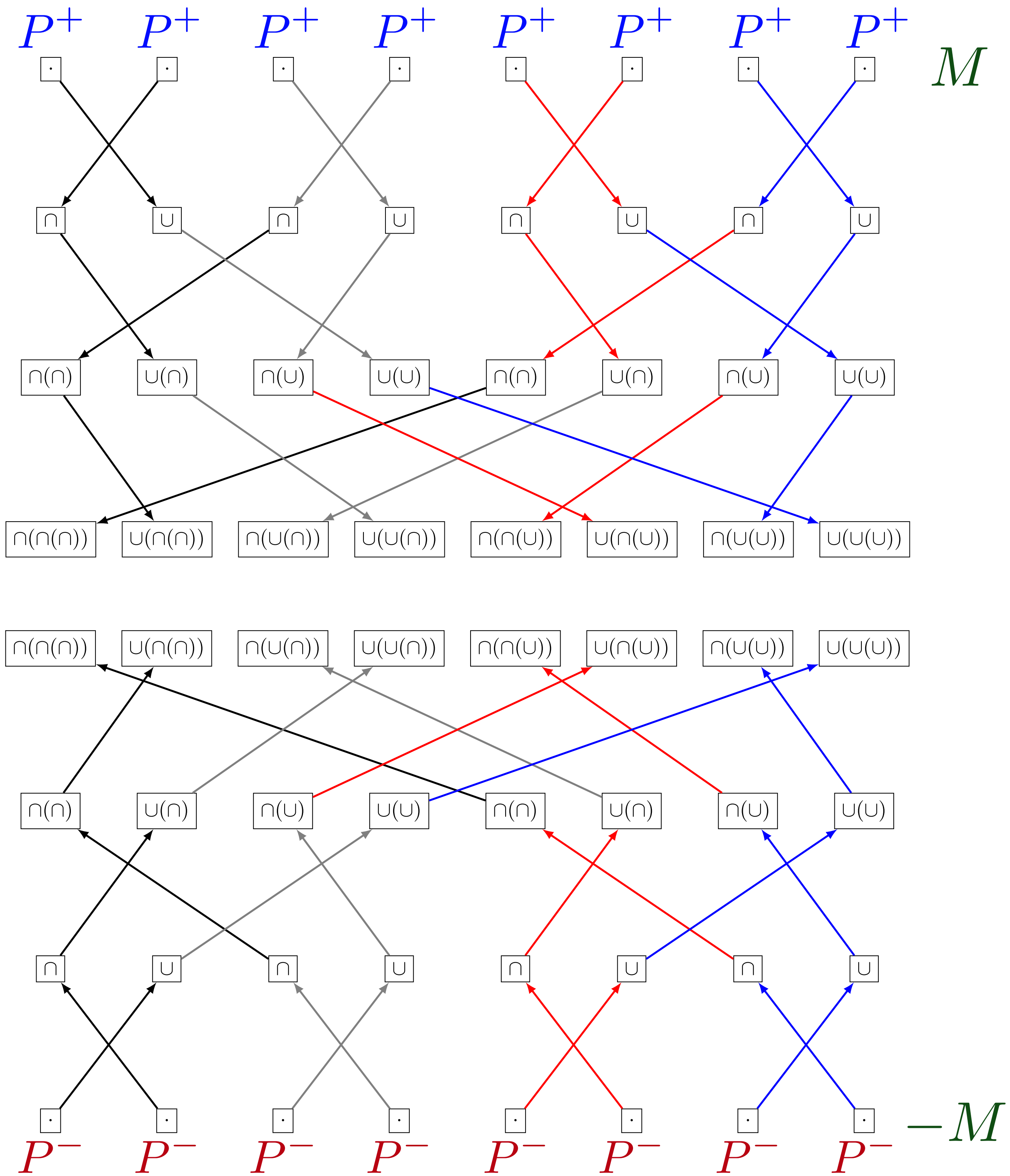
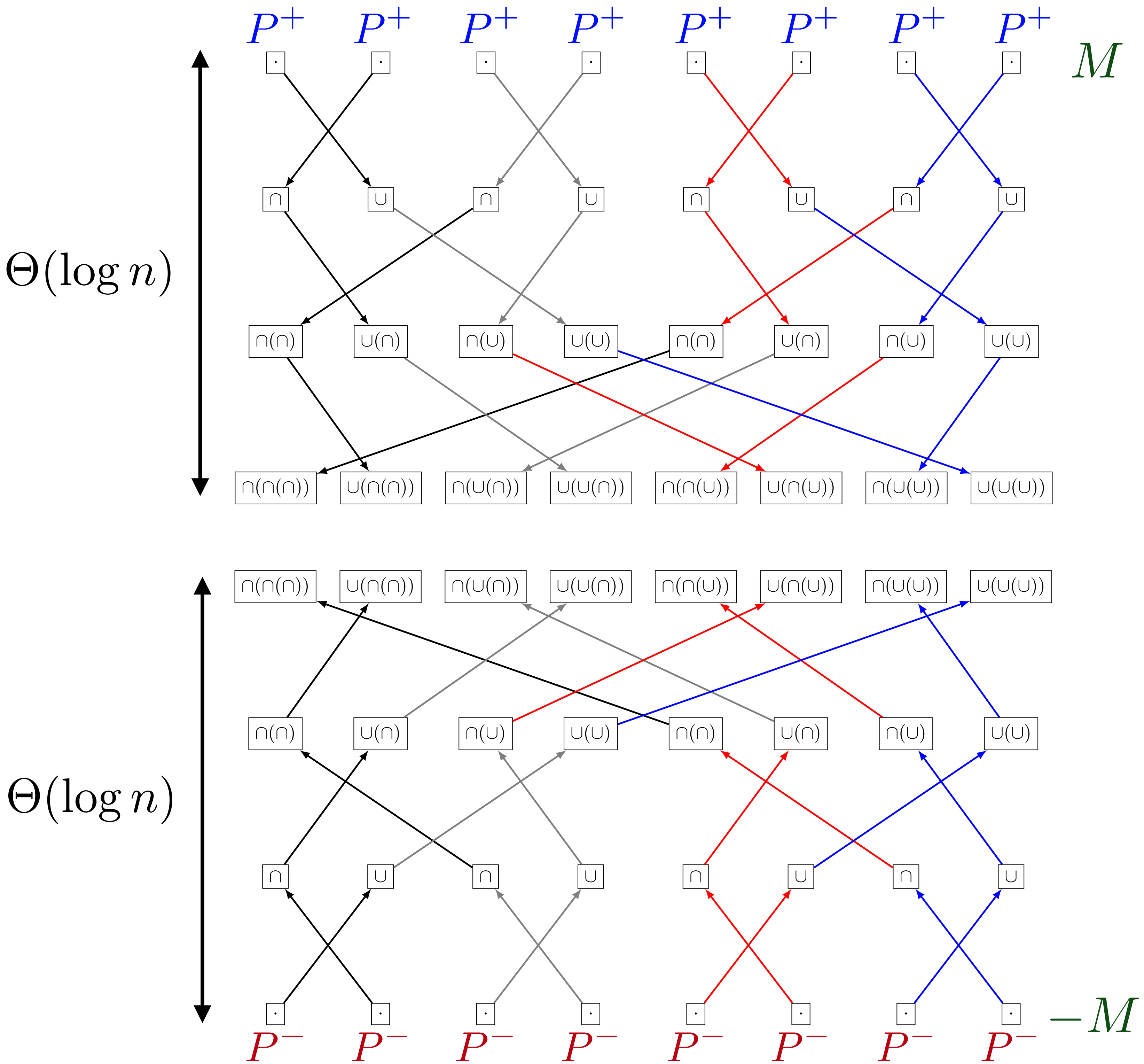


Bounding M



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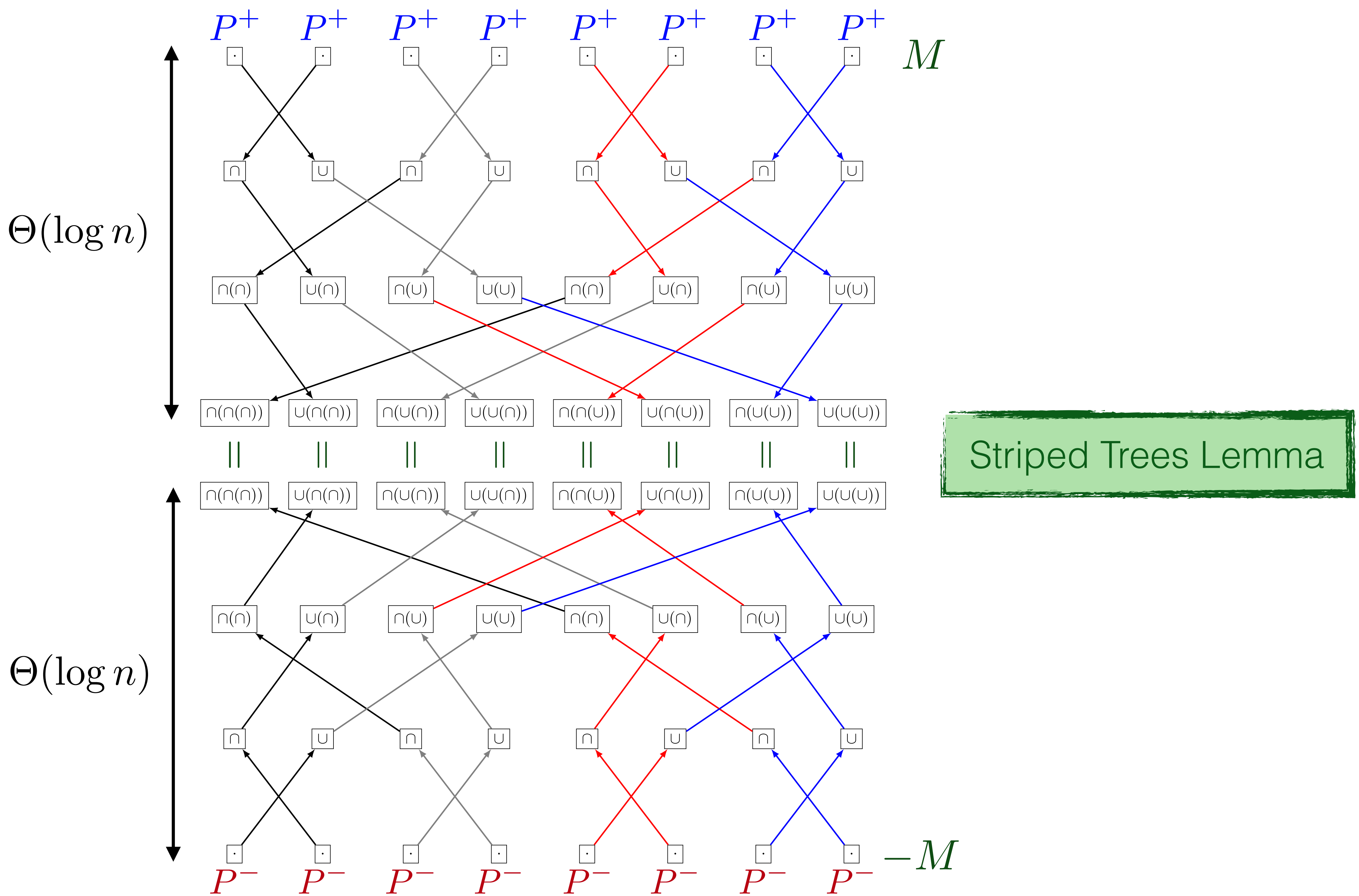
Bounding M



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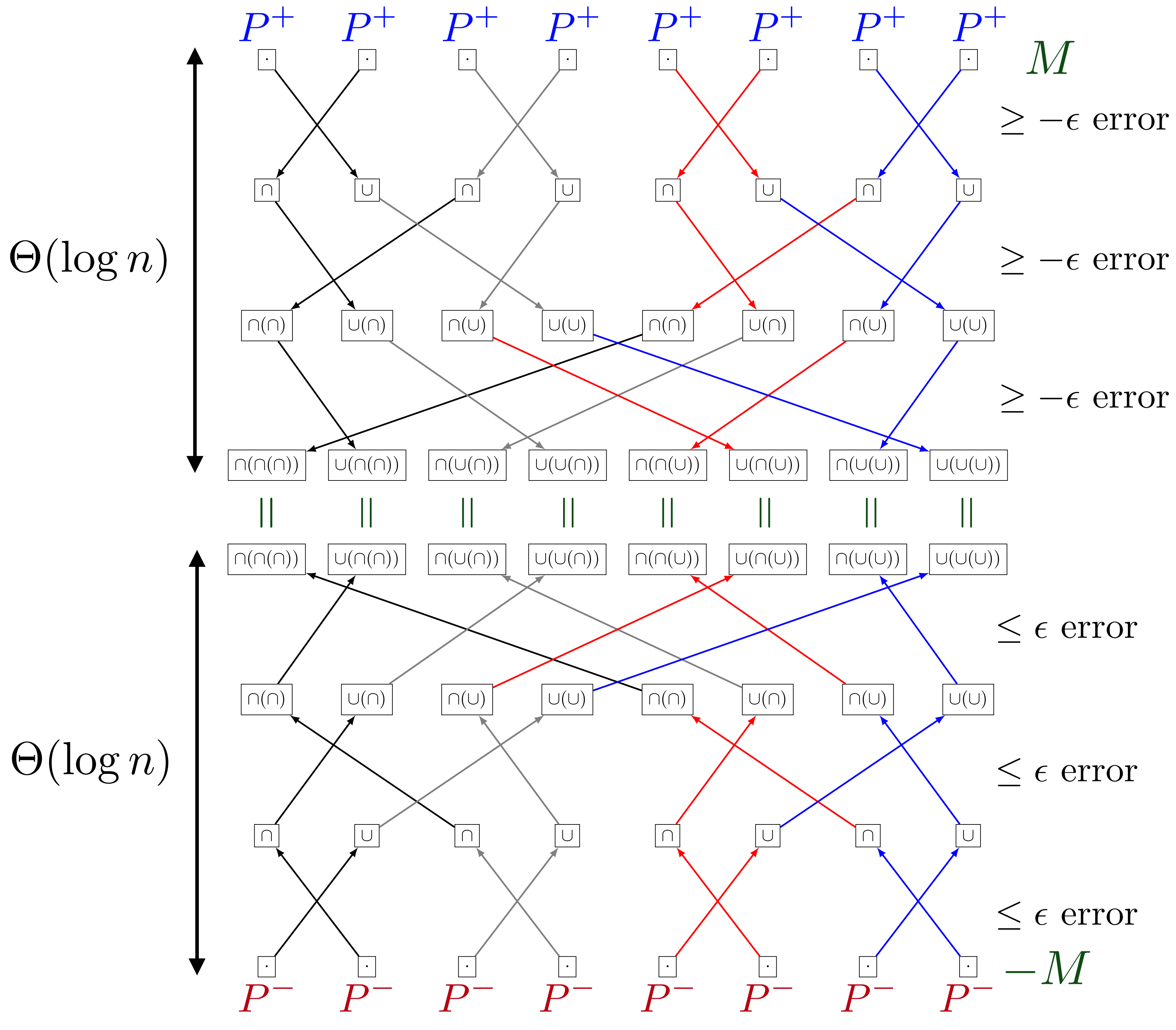
Bounding M



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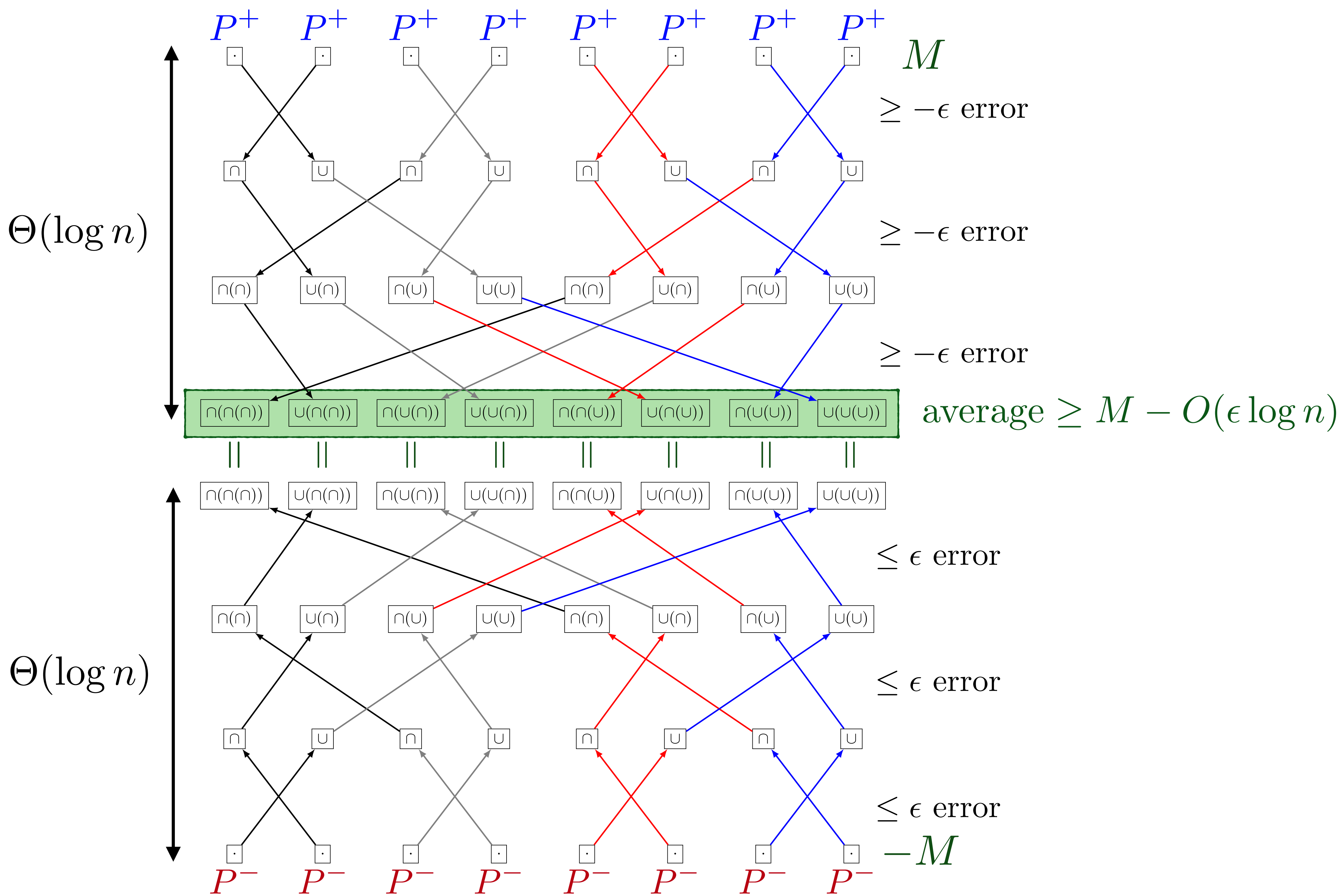
Bounding M



P+ P+ P+ P+ P+ P+ P+ P+
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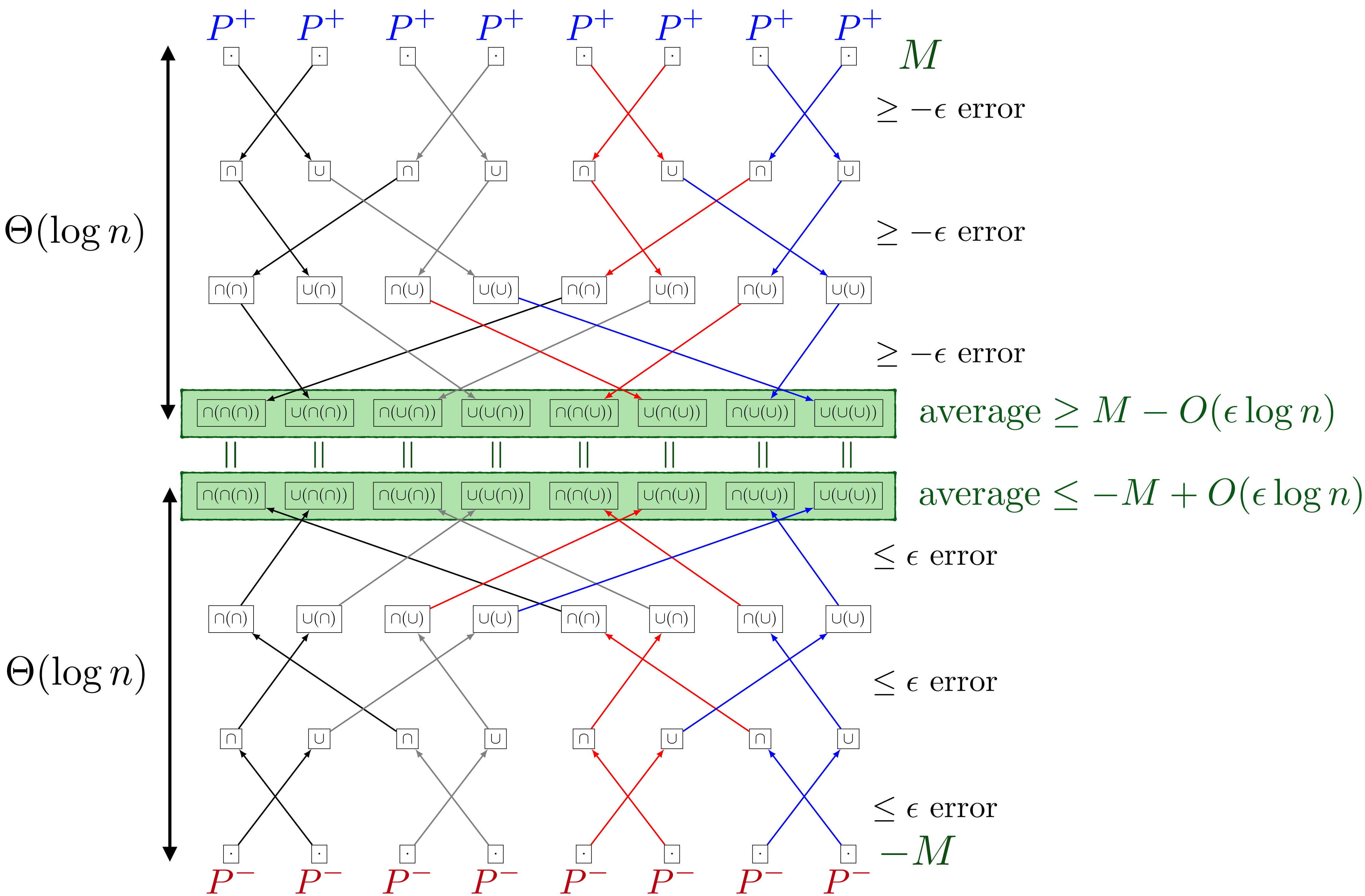
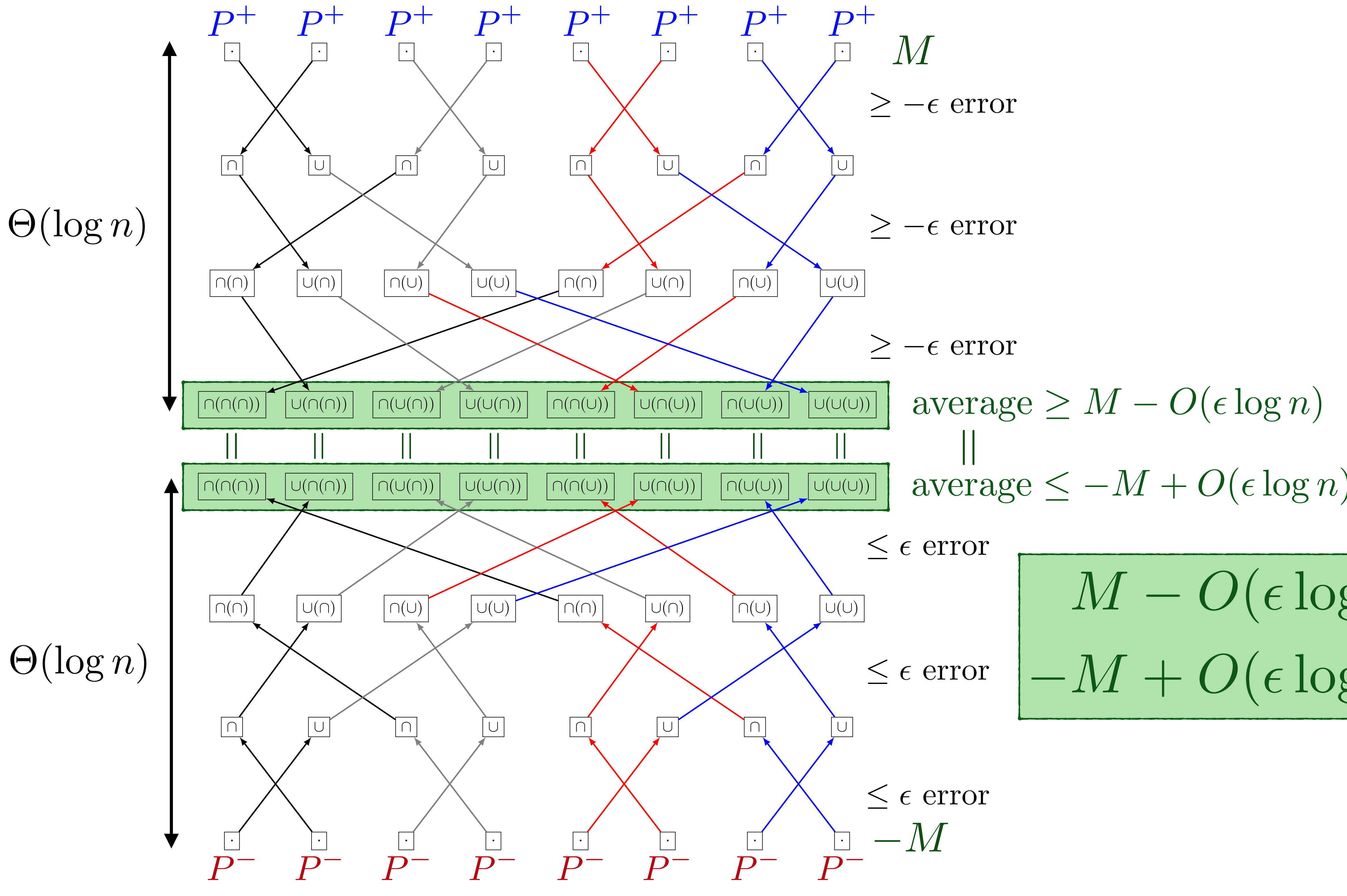
≥ −ϵ error
≥ −ϵ error
Θ(log n)
∩ ∪ ∩ ∪ ∩ ∪ ∩ ∪
∩(∩) ∪(∩) ∩(∪) ∪(∪) ∩(∩) ∪(∩) ∩(∪) ∪(∪)
∩(∩(∩)) ∪(∩(∩)) ∩(∪(∩)) ∪(∪(∩)) ∩(∩(∪)) ∪(∩(∪)) ∩(∪(∪)) ∪(∪(∪))
average ≥ M − O(ϵ log n)
∩(∩(∩)) ∪(∩(∩)) ∩(∪(∩)) ∪(∪(∩)) ∩(∩(∪)) ∪(∩(∪)) ∩(∪(∪)) ∪(∪(∪))
≤ ϵ error
∩(∩) ∪(∩) ∩(∪) ∪(∪) ∩(∩) ∪(∩) ∩(∪) ∪(∪)
≤ ϵ error
Θ(log n)
∩ ∪ ∩ ∪ ∩ ∪ ∩ ∪
≤ ϵ error
−M
P− P− P− P− P− P− P− P−

Bounding M



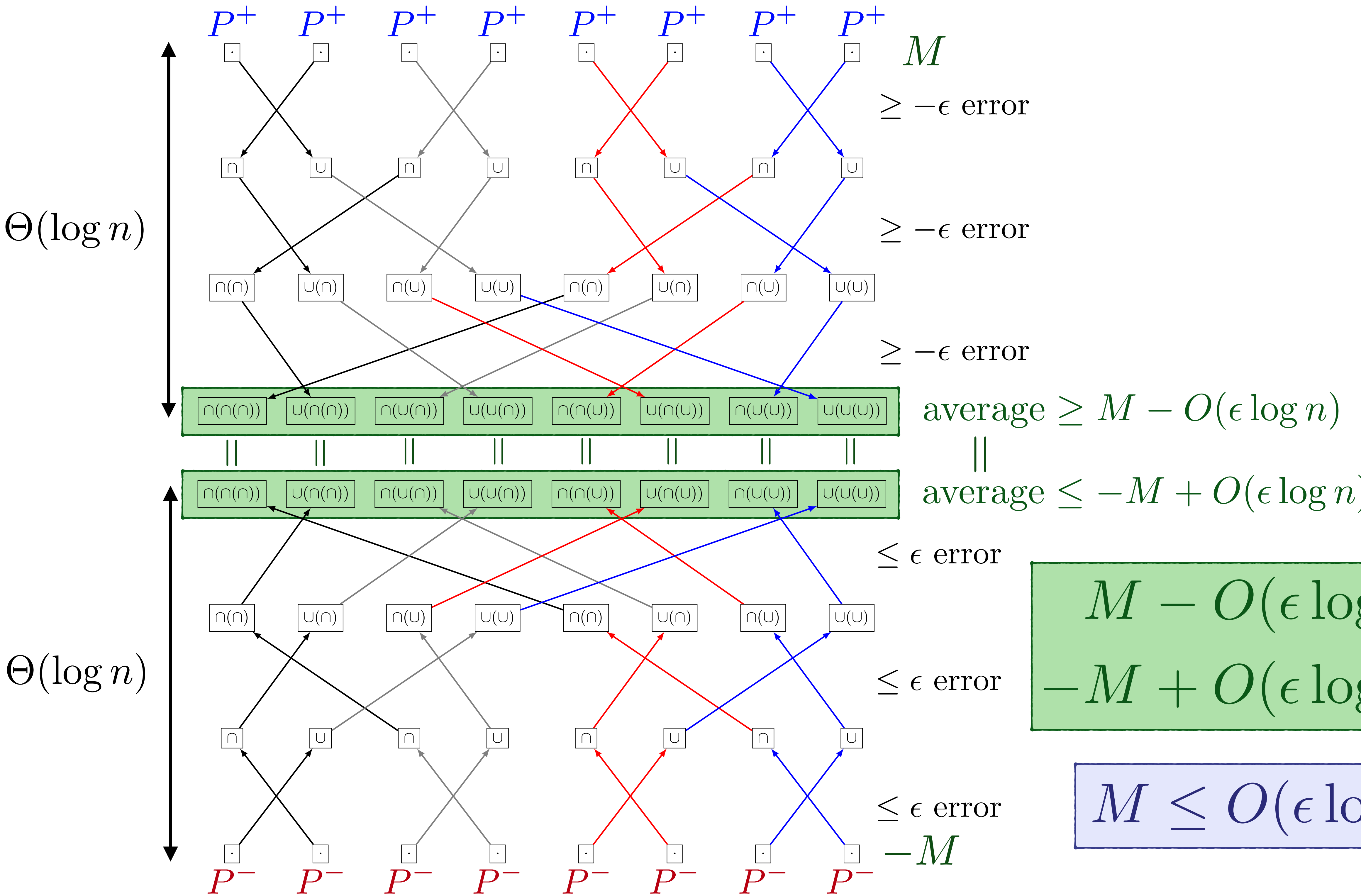
P+ P+ P+ P+ P+ P+ P+ P+
M
≥ −ϵ error
≥ −ϵ error
≥ −ϵ error
Θ(log n)
average ≥ M − O(ϵ log n)
average ≤ −M + O(ϵ log n)
≤ ϵ error
≤ ϵ error
≤ ϵ error
Θ(log n)
−M
P− P− P− P− P− P− P− P−

Bounding M



$$M - O(\epsilon \log n) \leq -M + O(\epsilon \log n)$$

Bounding M



$$M - O(\epsilon \log n) \leq -M + O(\epsilon \log n)$$

$$M \leq O(\epsilon \log n)$$

Closeness to Modularity

- …thus, the maximum value of an ϵ-approximately modular function *F* on *[n]* (that is best approximated by the all-0 function *z*) is $O(\epsilon \cdot \log n)$.

Closeness to Modularity

- …thus, the maximum value of an ϵ-approximately modular function *F* on *[n]* (that is best approximated by the all-0 function *z*) is $O(\epsilon \cdot \log n)$.

- It follows that, if *f* is an ϵ-approximately modular function on *[n]*, there exists a modular function *g* such that

$$\forall S \subseteq [n] \quad |f(S) - g(S)| \leq O(\epsilon \cdot \log n)$$

Conclusion

- We have studied
  - the polynomial-time approximability, and
  - the Borsuk-Ulam approximability,

  of *approximately modular functions.*

Conclusion

- We have studied
  - the polynomial-time approximability, and
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  of *approximately modular functions.*
- **Open questions**
  - Is our logarithmic upper bound on the distance to modularity tight?
  - What happens for functions that are (additively) approximately sub-modular?

Thanks!