Balanced implementability of Sequencing Rules

Parikshit De
Manipushpak Mitra
Indian Statistical Institute
Kolkata

IISc
January, 2016.

Examples

- Students wanting to use copier machine or computer in a department.

- Amateur astronomers willing to use public telescope.

- Cars in a repair and maintenance store for servicing.

- Patients in a doctor's clinic for treatment.

- People from a locality arriving to a minister to report and resolve their problems.

Sequencing Problem

- Set of agents $N = \{1, \ldots, n\}$ and a single facility.

- Let $\forall i \in N$, $s_i \in \Re_{++}$ where s_i denotes the processing time of ith agent.

- $\theta_i \in \Re_{++}$, the cost of waiting per unit of time.

- A profile is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Re^n_{++}$.

- An allocation rule is a mapping $\sigma : \Re^n_{++} \to \Sigma(N)$ that specifies for each $\theta \in \Re^n_{++}$ an allocation(rank) vector $\sigma(\theta) \in \Sigma(N)$.

- Agent i's position is denoted by $\sigma_i(\theta)$ which is an input of the vector $\sigma(\theta)$.

- Given $\sigma(\theta) \in \Sigma(N), \forall i \in N$,
$P_i(\sigma(\theta)) = \{j \in N | \sigma_j(\theta) < \sigma_i(\theta)\}$ denotes the set of predecessors of i and similarly
$P'_i(\sigma(\theta)) = \{j \in N | \sigma_j(\theta) > \sigma_i(\theta)\}$ denotes the set of successors of i.

- Utility function of each agent $i \in N$ is quasi-liner and is of the form $U_i(\sigma(\theta), \tau_i(\theta); \theta_i) = -S_i(\sigma(\theta))\theta_i + \tau_i(\theta)$ where the job completion time is $S_i(\sigma(\theta)) = s_i + \sum_{j \in P_i(\sigma(\theta))} s_j$.

- A transfer rule is a mapping $\tau : \Re^n_{++} \to \Re^n$ that specifies for each profile $\theta \in \Re^n_{++}$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \Re^n$.

- A direct mechanism (σ, τ) constitutes of an allocation rule σ and a transfer rule τ.

Allocation Rules

- Outcome Efficiency:

$$\sigma^e(\theta) \in \arg\min_{\sigma \in \Sigma(N)} \sum_{i \in N} \theta_i S_i(\sigma)$$

$$\theta_i/s_i > \theta_j/s_j \quad iff \;\; \sigma_i^e(\theta) < \sigma_j^e(\theta)$$

- Rawlsian Fairness:

$$\sigma^r(\theta) \in \arg\min_{\sigma \in \Sigma(N)} \max\{\theta_i S_i(\sigma)\}_{i \in N}$$

$$If \;\; \theta_i > \theta_j, \;\; then \;\; \sigma_i^r(\theta) < \sigma_j^r(\theta).$$

Implementability Criterion

Definition 1 A mechanism (σ, τ) *implements* the sequencing rule σ in dominant strategies if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta_i' \in \Theta$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$,

$$U_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq U_i(\sigma(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i). \tag{1}$$

$$-S_i(\sigma(\theta))\theta_i + \tau_i(\theta) \geq -S_i(\sigma(\theta_i', \theta_{-i}))\theta_i + \tau_i(\theta_i', \theta_{-i})$$

Definition 2 A rule σ satisfies *non-increasingness* (or NI) if for any $i \in N$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the chosen order $\sigma(\theta_i, \theta_{-i})$ for each $\theta_i \in \Theta$ is such that the job completion time $S_i(\sigma(\theta_i, \theta_{-i}))$ is non-increasing in θ_i.

Proposition 1 A sequencing rule σ is implementable if and only if it is an NI sequencing rule.

The outcome efficient rule and the Rawlsian Fairness rule are implementable.

Robert's Affine Maximization Theorem

- A classic result in mechanism design in a quasi-linear set-up is the Roberts' affine maximizer theorem (see Roberts (1979)) for multidimensional type spaces with finite set of alternatives.

- Roberts (1979) showed that if there are at least three alternatives and the type space is unrestricted, then every onto implementable allocation rule is an affine maximizer.

Affine cost minimization sequencing rules

Definition 3 A sequencing rule $\sigma^{w,\kappa} : \Theta^n \rightarrow \Sigma(N)$ is an *affine cost minimizer* (ACM) if for each $\theta \in \Theta^n$,

$$\sigma^{w,\kappa}(\theta) \in \arg \min_{\sigma \in \Sigma'(N)} \left\{ \kappa(\sigma) + \sum_{j \in N} w_j \theta_j S_j(\sigma) \right\},$$

where $\Sigma'(N) \subseteq \Sigma(N)$, $w_j \geq 0$ for all $j \in N$ and $\kappa : \Sigma'(N) \rightarrow \mathbb{R}$.

Proposition 2 For any Ω_N^s,

$$ACM(\Omega_N^s) \subseteq NI(\Omega_N^s) \text{ and } ACM(\Omega_N^s) \neq NI(\Omega_N^s).$$

Example 1 Consider any sequencing problem Ω_N^s with $|N| = 2$. Define the sequencing rule σ^V that, given any two positive numbers a_1 and a_2 satisfies the following:

1. For any profile $\theta = (\theta_1, \theta_2)$ such that $\theta_1 < a_1$ and $\theta_2 > a_2$, $\sigma^V(\theta) = (\sigma_1^V(\theta) = 2, \sigma_2^V(\theta) = 1)$.

2. For all other profiles $\theta' = (\theta'_1, \theta'_2)$ such that either $\theta'_1 \geq a_1$ or $\theta'_2 \leq a_2$, $\sigma^V(\theta') = (\sigma_1^V(\theta') = 1, \sigma_2^V(\theta') = 2)$.

Example 2 Consider any sequencing problem Ω_N^s with $|N| \geq 3$. Define σ^{NA} that satisfies the following properties:

1. For any θ s.t. $\theta_1/s_1 \geq \min_{j \in N \setminus \{1\}}(\theta_j/s_j)$. Then $\sigma^{NA}(\theta)$ specifies that $1 = \sigma_1^{NA}(\theta) < \sigma_j^{NA}(\theta)$ for any $j \in N \setminus \{1\}$, and, for any $j, k \in N \setminus \{1\}$, $\sigma_j^{NA}(\theta) \leq \sigma_j^{NA}(\theta)$ if and only if $(\theta_j/s_j) \geq (\theta_k/s_k)$.

2. For any θ' s.t. $\theta'_1/s_1 < \min_{j \in N \setminus \{1\}}(\theta'_j/s_j)$. Then $\sigma^{NA}(\theta')$ specifies that $n = \sigma_1^{NA}(\theta') > \sigma_j^{NA}(\theta')$ for any $j \in N \setminus \{1\}$, and, for any $j, k \in N \setminus \{1\}$, $\sigma_j^{NA}(\theta') \leq \sigma_k^{NA}(\theta')$ if and only if $(\theta'_j/s_j) \geq (\theta'_k/s_k)$.

It is quite easy to see that the sequencing rule σ^{NA} satisfies NI. That σ^{NA} is not an affine cost minimizer will be established in the next theorem.

Affine cost minimization:Continued

Consider the special case where $\Sigma'(N) = \Sigma(N)$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma(N)$.

Definition 4 A sequencing rule $\sigma^w : \Theta^n \to \Sigma(N)$ is a *strong affine cost minimizer* (SACM) if for each $\theta \in \Theta^n$,

$$\sigma^w(\theta) \in \arg \min_{\sigma \in \Sigma(N)} \sum_{j \in N} w_j \theta_j S_j(\sigma),$$

where $w_j \geq 0$ for all $j \in N$.

$$(w_i\theta_i)/s_i > (w_j\theta_j)/s_j \quad iff \;\; \sigma_i^w(\theta) < \sigma_j^w(\theta)$$

1. If $w_i = 1$ for all $i \in N$, then we have outcome efficiency.

2. If $w_i = s_i$ for all $i \in N$, then we have Rawlsian Fairness.

Cut-off based transfers

For each $i \in N$, we first select any function $h_i : \Theta^{|N \setminus \{i\}|} \rightarrow \mathbb{R}$ and then, given any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, we consider the cut-off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ where $0 := \theta_i^{(T)} < \theta_i^{(T-1)}(\theta_{-i}) < \ldots < \theta_i^{(2)}(\theta_{-i}) < \theta_i^{(1)}(\theta_{-i}) < \theta_i^{(0)} := \infty$. Given the selected function $h_i : \Theta^{|N \setminus \{i\}|} \rightarrow \mathbb{R}$, for any profile θ_{-i} of all but agent i and the associated cut-off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$, the transfer of agent i is the following:

(PI1) For any $\theta_i \in \Theta \setminus \{\theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i})\}$, $\tau_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i, \theta_{-i})$ where

$$I_i(\theta_i, \theta_{-i}) = \begin{cases} 0 & \text{if } \theta_i \in (\theta_i^{(T)}, \theta_i^{(T-1)}), \\ \sum_{r=t}^{T-1} \theta_i^{(r)} D_r & \text{if } \theta_i \in (\theta_i^{(t)}, \theta_i^{(t-1)}),\ t < T,\ T \geq 2. \end{cases} \tag{2}$$

(PI2) For $T \geq 2$, any $t \in \{1, \ldots, T-1\}$ and cut-off point $\theta_i^{(t)}(\theta_{-i})$, $\tau_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i^{(t)}, \theta_{-i})$ where the incentive payment is $I_i(\theta_i^{(t)}, \theta_{-i}) = I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)}(\theta_{-i})\overline{D}_t + \theta_i^{(t)} D_t$ and $\theta_i^t \in (\theta_i^{(t)}, \theta_i^{(t-1)})$.

Transfer of any $i \in N$ for the profile θ is

$$\tau_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i, \theta_{-i}),$$

where $I_i(\theta_i, \theta_{-i})$ is the incentive payment and $h_i(\theta_{-i})$ is the agent specific constant.

1. Outcome efficiency: For any $i \in N$ for the profile θ is

$$\tau_i^e(\theta_i, \theta_{-i}) = h_i^e(\theta_{-i}) - \sum_{j \in P_i'(\sigma^e(\theta))} \theta_j s_i,$$

where the incentive payment is given by

$$I_i^e(\theta_i, \theta_{-i}) = \sum_{j \in P_i'(\sigma^e(\theta))} \theta_j s_i.$$

2. Rawlsian Fairness: For any $i \in N$ for the profile θ is

$$\tau_i^r(\theta_i, \theta_{-i}) = h_i^r(\theta_{-i}) - \sum_{j \in P_i'(\sigma^r(\theta))} \theta_j s_j,$$

where the incentive payment is given by

$$I_i^r(\theta_i, \theta_{-i}) = \sum_{j \in P_i'(\sigma^r(\theta))} \theta_j s_j.$$

Balanced Implementability

Definition 5 A rule σ is *implementable with balanced transfers* if there exists a mechanism (σ, τ) that implements it with budget balancing transfers, that is, for all $\theta \in \Theta^n$, $\sum_{j \in N} \tau_j(\theta) = 0$.

$$\tau_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i, \theta_{-i}),$$

$$\sum_{j \in N} \tau_j(\theta) = \sum_{j \in N} h_j(\theta_{-j}) - \sum_{j \in N} I_j(\theta) = 0.$$

For each rofile θ, aggregate incentive payment is $(n-1)$ type separable.

$$I(\theta) = \sum_{j \in N} h_j(\theta_{-j}), \text{ where } I(\theta) := \sum_{j \in N} I_j(\theta).$$

Existing Results on Balanced Implementation

1. The outcome efficient rule is implementable with balanced transfers for $|N| \geq 3$.

2. The Rawlsian Fairness rule is implementable with balanced transfers for $|N| \geq 3$.

3. When $|N| = 2$, neither outcome efficiency nor Rawlsian Fairness is implementable with balanced transfers.

Two-agents case

Definition 6 Let $\Omega^{(s_1,s_2)}_{\{1,2\}}$ be a two-agent sequencing problem. A sequencing rule σ^{Tx} is a *two agent balancing* (TAB) sequencing rule if there exists an agent $k \in \{1,2\}$ such that any one of the following conditions hold.

(T1) There exists $a_k > 0$ such that $a_l = (a_k s_l)/s_k > 0$ and either $A_k(\sigma^{T1a}) = \{\theta \in \Theta^2 \mid either\ \theta_k \geq a_k\ or\ \theta_l \leq a_l\}$ or $A_k(\sigma^{T1b}) = \{\theta \in \Theta^2 \mid either\ \theta_k > a_k\ or\ \theta_l < a_l\}$ (see Figure 1 where we have (T1a) and (T1b) for $k = 1$).

(T2) There exists a real number $a_k > 0$ such that either $A_k(\sigma^{T2a}) = \{\theta \in \Theta^2 \mid \theta_k \geq a_k\}$ or $A_k(\sigma^{T2b}) = \{\theta \in \Theta^2 \mid \theta_k > a_k\}$ (See Figure 2 where we have (T2a) and (T2b) for $k = 1$).

Theorem 1 *A non-constant $\sigma \in NI(\Omega^{(s_1,s_2)}{}_{\{1,2\}})$ is implementable with balanced transfers if and only if it is a TAB sequencing rule σ^{Tx}.*

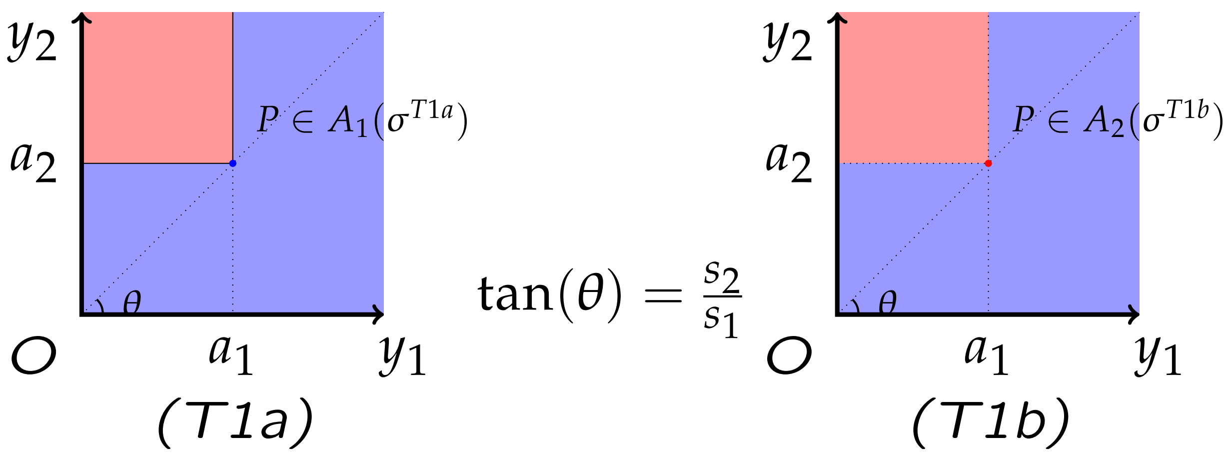
y_2
a_2
$P \in A_1(\sigma^{T1a})$
θ
O
a_1
y_1
(T1a)
$\tan(\theta) = \frac{s_2}{s_1}$
y_2
a_2
$P \in A_2(\sigma^{T1b})$
θ
O
a_1
y_1
(T1b)

Figure 1: (T1)

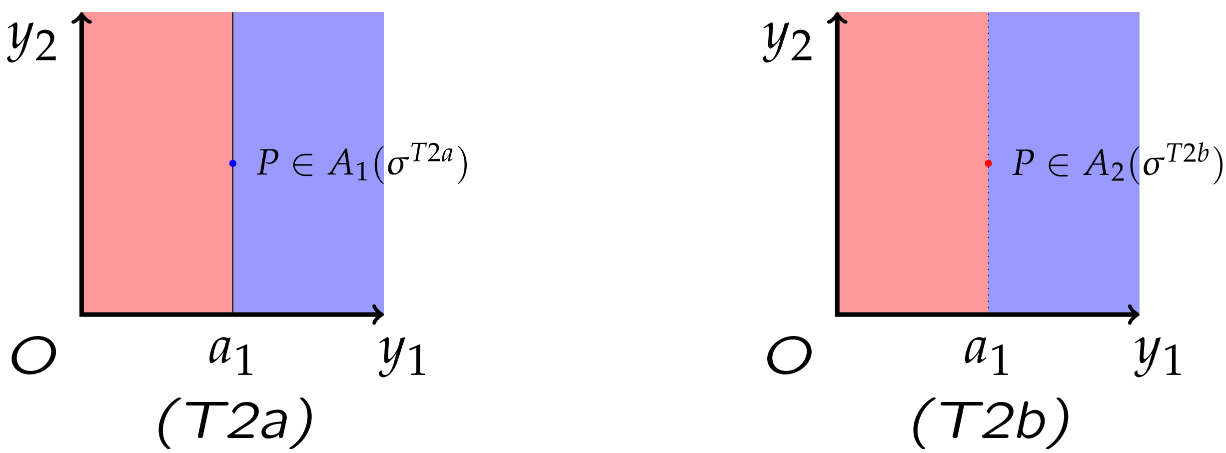

Figure 2: (T2)

More than two agents case

Consider any sequencing problem Ω_N^s with more than two agents and let Π_N be the set of all possible *priority* partitions of the agents where the order of representing the partition is important in terms of priority. Let $\pi(N) = (\pi_1, \ldots, \pi_K)$ $\in \Pi_N$ be any priority partition of the set of agents. The set of $\pi(N)$ induced orders is $\Sigma(\pi(N))$. Therefore, the set of priority partition $\pi(N)$ induced orders $\Sigma(\pi(N))$ are those orders where agents in π_1 are always served first, agents in π_2 are always served after agents in π_1 but before agents in π_3 (if any) and so on. If $K = 1$ so that $\pi(N) = (\pi_1 = \pi_K = \{N\})$, then $\Sigma(\pi(N)) = \Sigma(N)$ which is the set of all possible ordering on the set of agents N.

For example, for $\Pi_{\{1,2,3\}}$, there are four types of priority partitions. These are $\pi^c = (\pi_1 = \{i\}, \pi_2 = \{j\}, \pi_3 = \{k\})$, $\pi^{21} = (\pi_1 = \{i,j\}, \pi_2 = \{k\})$, $\pi^{12} = (\pi_1 = \{i\}, \pi_2^{12} = \{j,k\})$ and $\bar{\pi} = (\pi_1 = \{1,2,3\})$ where $i \neq j \neq k \neq i$. For π^c, $\Sigma(\pi^c) = \{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\}$, for π^{21}, $\Sigma(\pi^{21}) = \{\{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\} \cup \{(\sigma_i = 2, \sigma_j = 1, \sigma_k = 3)\}\}$, for π^{12}, $\Sigma(\pi^{12}) = \{\{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\} \cup \{(\sigma_i = 1, \sigma_j = 3, \sigma_k = 2)\}\}$ and finally for $\bar{\pi}$, we have the set of all possible orders on the set of agents, that is, $\Sigma(\bar{\pi}) = \Sigma(\{1,2,3\})$.

Definition 7 Consider any priority partition $\pi(N) \in \Pi_N$ and let $f = \{f_1, \ldots, f_n\}$ be a set of agent specific increasing and one-to-one functions $f_j : \Theta \to \mathbb{R}_+$. The sequencing rule $\sigma^{\pi(N),f} : \Theta^n \to \Sigma(N)$ satisfies *group priority based cost minimization* (GP-CM) if for each $\theta \in \Theta^n$,

$$\sigma^{\pi(N),f}(\theta) \in \arg \min_{\sigma \in \Sigma(\pi(N))} \sum_{j \in N} f_j(\theta_j) S_j(\sigma).$$

- For any $\pi(N) \in \Pi_N$, any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with the property that there exists an agent $j \in N$ such that $f_j(.)$ is non-linear is an NI sequencing rule which is not an ACM.

- For any $\pi(N) \in \Pi_N$, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ where $f_j(.)$ is linear for all $j \in N$ is an ACM sequencing rule. Specifically, any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_j > 0$ for all $j \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(N)$ and there exists a priority partition $\pi(N) \in \Pi_N$ such that $\Sigma'(N) = \Sigma(\pi(N))$ is a GP-CM sequencing rule.

- The GP-CM sequencing rule is not onto for any $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ such that $K \geq 2$ since, in that case, $\Sigma(N) \setminus \Sigma(\pi(N)) \neq \emptyset$ and any order $\sigma \in \Sigma(N) \setminus \Sigma(\pi(N))$ is never chosen.

- For $\pi(N) \in \Pi_N$ such that $K = 1$ so that the $\pi(N) = (\pi_1 = \pi_K = \{N\})$ is the grand coalition, $\Sigma(\pi(N)) = \Sigma(N)$ and any such GP-CM $\sigma^{\pi(N),f}$ is onto.

- A GP-CM sequencing rule $\sigma^{\pi(N),f}$ is a constant sequencing rule if $\pi(N) = (\pi_1, \ldots, \pi_K)$ is such that $K = n$.

- A GP-CM sequencing rule $\sigma^{\pi(N),f}$ gives the outcome efficient sequencing rule σ^* if $\pi(N) = (\{N\})$ and $f_j(\theta_j) = \theta_j$ for all $j \in N$.

- A GP-CM sequencing rule $\sigma^{\pi(N),f}$ gives the Rawlsian Fairness sequencing rule $\tilde{\sigma}$ if $\pi(N) = (\{N\})$ and $f_j(\theta_j) = s_j\theta_j$ for all $j \in N$.

For any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with the priority partition $\pi(N) \in \Pi_N$, *modified urgency index* $f_j(\theta_j)/s_j$ is used to determine the profile contingent order of serving the agents. Specifically, like Smith's (1956) rule for outcome efficient sequencing rule σ^*, for any GP-CM $\sigma^{\pi(N),f}$, the selected order $\sigma^{\pi(N),f}(\theta)$ satisfies the following condition.

(GP-CM) For any $i, j \in \pi_k \in \pi(N)$,
$$(f_i(\theta_i)/s_i) \geq (f_j(\theta_j)/s_j) \Leftrightarrow \sigma_i^{\pi(N),f}(\theta) \leq \sigma_j^{\pi(N),f}(\theta).$$

Given the tie-breaking rule, this profile contingent selection $\sigma^{\pi(N),f}(\theta)$ is unique.

Theorem 2 Consider any sequencing problem Ω_N^s with more than two agents. For any priority partition $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ and for any given set of functions $f = \{f_1, \ldots, f_n\}$ that are increasing and onto, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfers.