Robust Sequential Attacker-Defender Game with Redeployment

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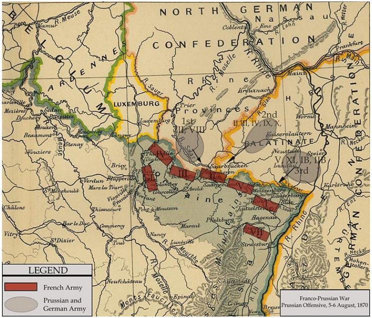
The classical Colonel Blotto game is one of the earliest strategic game used to model conflicts in multiple battlefields



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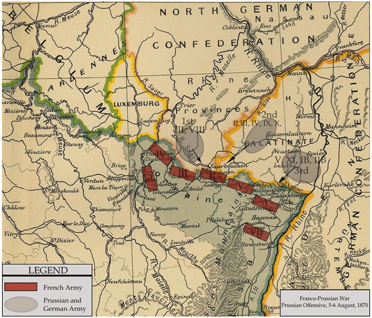

- Two-person zero-sum game where two players simultaneously allocate limited resources to multiple battlefields
- The player who allocate more resources in one battlefield wins, or has higher winning probability for that particular battlefield.

Example

Classical model: Each player has 100 troops for 3 fronts

- whoever allocates the most troops to a front will win the front
- the players do not know how their opponents will distribute troops
- players want to win as many fronts as possible.

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Players	Front 1	Front 2	Front 3
1	100/3	100/3	100/3
2	50	50	0

Player 2 wins 2 fronts while Player 1 wins 1 front

Equilibrium Strategy

No pure equilibrium strategy

Players	Front 1	FRont 2	Front 3
1	51	48	1
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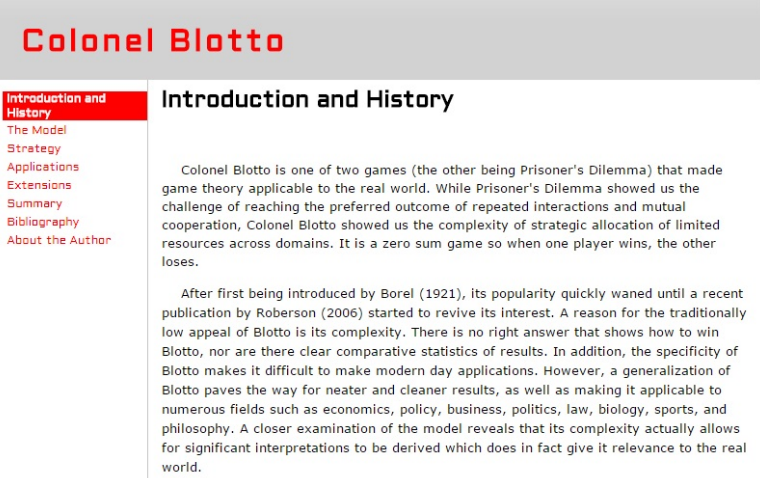
Proposition [Gross and Wagner (1950)] The Colonel Blotto game has a mixed strategy equilibrium in which the marginal distributions are uniform on $[0, 2N/n]$ along all fronts.

Construction of the joint equilibrium strategy is still not completely solved

Applications

https://sites.google.com/site/briankimblotto/home



Colonel Blotto

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Introduction and History

Colonel Blotto is one of two games (the other being Prisoner's Dilemma) that made game theory applicable to the real world. While Prisoner's Dilemma showed us the challenge of reaching the preferred outcome of repeated interactions and mutual cooperation, Colonel Blotto showed us the complexity of strategic allocation of limited resources across domains. It is a zero sum game so when one player wins, the other loses.

After first being introduced by Borel (1921), its popularity quickly waned until a recent publication by Roberson (2006) started to revive its interest. A reason for the traditionally low appeal of Blotto is its complexity. There is no right answer that shows how to win Blotto, nor are there clear comparative statistics of results. In addition, the specificity of Blotto makes it difficult to make modern day applications. However, a generalization of Blotto paves the way for neater and cleaner results, as well as making it applicable to numerous fields such as economics, policy, business, politics, law, biology, sports, and philosophy. A closer examination of the model reveals that its complexity actually allows for significant interpretations to be derived which does in fact give it relevance to the real world.

Real Play

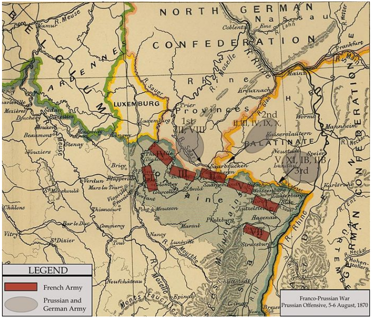
http://www1.maths.leeds.ac.uk/ pmt6jrp/personal/blotto.html
Colonel Blotto competition run in January 1990

Player Number	Disposition of forces........										W	D	L	pts
26	17	3	17	3	17	3	17	3	17	3	64	33	9	161
80	19	1	1	19	19	1	1	19	19	1	56	44	6	156
27	19	2	16	18	3	19	0	3	17	3	61	33	12	155
39	1	19	1	1	19	19	19	1	19	1	54	46	6	154
24	7	18	18	2	9	3	18	5	2	18	61	30	15	152
82	2	10	1	18	19	3	20	2	8	17	61	26	19	148
3	17	0	17	0	17	0	16	0	17	16	64	19	23	147
7	17	0	17	0	16	0	16	17	16	1	64	17	25	145
22	16	17	5	2	4	19	1	18	15	3	57	31	18	145
38	16	0	17	16	0	0	17	17	0	17	64	17	25	145
43	0	0	0	0	17	17	17	17	16	16	61	23	22	145
4	17	15	0	0	17	17	17	0	0	17	59	26	21	144
66	1	17	1	1	21	19	17	1	1	21	50	41	15	141
53	2	4	8	16	16	20	15	0	15	4	54	32	20	140
35	6	6	16	6	16	16	6	16	6	6	55	29	22	139
17	5	10	18	16	1	5	18	16	10	1	56	26	24	138
63	1	1	15	1	18	18	10	2	17	17	52	32	22	136
11	4	16	0	0	16	16	16	0	16	16	58	19	29	135

Motivations

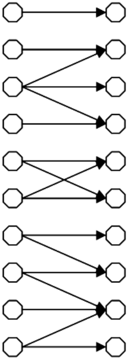
Attacker-Defender Model - Terrorists attacks

- Attacker chooses timing of attacks based on defender deployment
- Defender's ability to re-deploy forces crucial in the outcome



Question

Suppose troops on some fronts can be re-deployed to other fronts based on the attacker's strategy



What is the optimal strategy for defender and attacker?

Engagement Rules

Contest Success Function (CSF)

- Classical Blotto's CSF is not continuous on forces deployment
- Lottery Rule:

$$\sum_{i=1}^{n} \frac{d_i^r}{d_i^r + a_i^r}$$

leads however to simple pure equilibrium strategy

We modify the winning probability of defender on one front to be

$$\begin{cases} \frac{d_i}{a_i}, & \textbf{if } d_i \leq a_i, a_i \neq 0 \\ 1, & \textbf{if } d_i > a_i, \textbf{or } a_i = 0 \end{cases}$$

Implications for Classical Blotto

- Suppose the attacker's resources is k times the resources of the defender (D).
- Number of fronts $= N$

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(2). When $k \geq 2$, attacker will attack all the fronts and evenly allocates its troops to each front, i.e. $a_i = \frac{kD}{N}, \forall i = 1, ..., N$.

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(2). When $k \geq 2$, attacker will attack all the fronts and evenly allocates its troops to each front, i.e. $a_i = \frac{kD}{N}, \forall i = 1, ..., N$.
When $k < 2$, attacker will only attack $M = int(\frac{kN}{2})$ number of fronts (where *int* stands for the nearest integer value), and evenly distribute its troops across those fronts, $a_i = \frac{kD}{M}, \forall i = 1, ..., N$.
(Mixed strategy)

Value of Flexibility: $k = 1$

In the Blotto Game, when $k = 1$, the value of the game to defender is $0.75N$:
Attacker chooses half the fronts to attack randomly and deploy equal number of forces on each front.

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What can the defender do to increase this value if it has the option to re-deploy troops?

If it has **full flexibility**, then the defender can win ALL the battle fronts with redeployment option.

Value under full flexibility $= N$

Value of Flexibility: $k = 1$

The defender do not need to have too much flexibility to attain a value close to N!

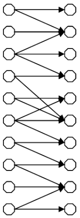
- Given a bipartite graph $\mathcal{G}(\mathcal{D}, \mathcal{A}, \mathcal{E})$, for any subset $S \subseteq \mathcal{A}$, its neighbour is defined as $\Gamma(S) = \{i \in \mathcal{D} | (i,j) \in \mathcal{E}, j \in S\}$.
- Given a bipartite graph $\mathcal{G}(\mathcal{D}, \mathcal{A}, \mathcal{E})$ is a $(\alpha, \lambda, \Delta)$- Expander if $deg(v) \leq \Delta$ for every node $v \in \mathcal{A}$. For any subset $S \subseteq \mathcal{A}$ with $|S| \leq \alpha|\mathcal{A}|$, its neighbour size $|\Gamma(S)| \geq \lambda|S|$.

An $(\alpha, \lambda, \Delta)$- Expander ensures that for a small subset S with $|S| \leq \alpha|\mathcal{A}|$, its neighbour is large enough to defend the forces in S.

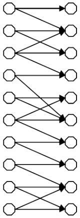
Defender can achive a value of $(1 - \frac{1}{\lambda})N$ under an expander structure.

Question

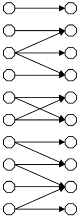
Which of the following is better for the defender?



: Asymmetric graph example 1



: Asymmetric graph example 2



: Asymmetric graph example 3

Figure: Asymmetric redeployment examples

3-Stage Model

(Discretization) We assume there is a finite set of strategies attacker can use, i.e. the number of troops attacker can locate to each front are in a finite scenario set

$$S = \{a_k, k = 1, ..., p | a_1 = 0\}$$

and

$$\hat{S} = \{\hat{a}_k | \hat{a}_1 = 0, \hat{a}_k = \frac{1}{a_k}, \forall k \neq 1\}$$

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In the **third stage**, defender can redeploy its troop given both parties' allocation in the first and second stage.

$$\begin{aligned} f_3(\boldsymbol{d}, \boldsymbol{y}) = \quad & \max \quad \textstyle\sum_j \left(\sum_{i:(i,j)\in\mathcal{G}} x_{ij} \sum_k y_{kj} \hat{a}_k \right) + \sum_j y_{1j} \\ & \text{s. t.} \quad \textstyle\sum_i x_{ij} \leq \sum_k y_{kj} a_k, && \forall j : (i,j) \in \mathcal{G} \\ & \quad \textstyle\sum_j x_{ij} \leq d_i, && \forall i : (i,j) \in \mathcal{G} \\ & \quad x_{ij} \geq 0, && \forall i,j \end{aligned}$$

3-Stage Model

In the **second stage**, attacker is to solve

$$Stage - 2 : f_2(\boldsymbol{d}) = \min_{\tilde{y}} f_3(\boldsymbol{d}, \tilde{\boldsymbol{y}}) \tag{1}$$

with $\tilde{y}_{kj}$ lying in the support of

$$\{\tilde{y}_{kj} | \sum_j \sum_k \tilde{y}_{jk} a_k \leq kD, \sum_k y_{kj} = 1, \forall j, y_{kj} \in \{0, 1\}\}.$$

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In the **first stage**, the defender is to maximize the concave function $f_2(\boldsymbol{d})$ subjective to the total troops budget constraint. Hence, the defender has a pure equilibrium strategy in the first stage.

Defender's Model

Therefore, the defender is to solve

$$Stage - 1 : f_1 = \max_{\sum_i d_i = D} f_2(\boldsymbol{d}) \tag{2}$$

The dual for Stage-2 problem is a MIQP -

$$\begin{aligned}
f_2(\boldsymbol{d}) = \quad \min \quad & \sum_j \sum_k \tilde{y}_{kj} a_k \alpha_j + \sum_i d_i \beta_i + \sum_j \tilde{y}_{1j} \\
\text{s. t.} \quad & \alpha_j + \beta_i - \sum_k \tilde{y}_{kj} \hat{a}_k - s_{ij} = 0 && \forall (i,j) \in \mathcal{G} \\
& \sum_j \sum_k \tilde{y}_{jk} a_k + t = kD \\
& \sum_k \tilde{y}_{kj} = 1, && \forall j \\
& \alpha_j, \beta_i \geq 0, && \forall i, j \\
& \tilde{y}_{kj} \in \{0, 1\}
\end{aligned} \tag{3}$$

Copositive and Completely Positive Cones

A completely positive cone is defined as

$$\mathcal{CP}_n := \{M \in S_n | \exists V \in \mathcal{R}_+^{n\times m}, \text{such that } M = VV^T\}$$

$$:= \{M \in S_n | \exists \boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k \in \mathcal{R}_+^n, \text{such that } M = \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\}$$

where S_n is $n \times n$ symmetric matrices.

Its dual, called copositive cone, is defined as

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Burer (2009) showed that the well-known $\mathcal{NP}$-hard problem, nonconvex quadratic problems with a mixture of binary and continuous variables has an equivalent completely positive formulation.

Completely Positive Cone

Under mild technical conditions, all binary MIQP problems can be reformulated as a (convex) conic program through lifting:

$$\mathbf{X_{ij}} = \mathbf{x_i x_j}$$

Theorem (Sam Burer)

$$\begin{aligned} Z_P = \max \quad & \sum_{j=1}^{n} Q_{ij} \mathbf{X_{i,j}} \\ s.t. \quad & \mathbf{a_i^T X a_i} - \mathbf{2 b_i a_i^T x} + \mathbf{b_i^2} = \mathbf{0}, \ \forall \mathbf{i} = \mathbf{1}, \ldots, \mathbf{m} \\ & X_{j,j} = x_j, \ \forall j \in \mathcal{B} \\ & \begin{pmatrix} 1 & \mathbf{x}^\mathbf{T} \\ \mathbf{x} & X \end{pmatrix} \succeq_{cp} 0 \end{aligned}$$

The CP and COP problems can be solved via SDP relaxation.

Approximating Choice Probability: Theory of Moments

Consider the following stochastic optimization problem:

$$Z_P := \sup_{\tilde{\mathbf{c}} \sim (\mu, \mathbf{\Sigma})^+} \mathbf{E}\left[Z(\tilde{\mathbf{c}})\right],$$

where $\tilde{\mathbf{c}} \sim (\mu, \mathbf{\Sigma})^+$ means

$$\tilde{\mathbf{c}} \in \{\tilde{\mathbf{X}} : \mathbf{E}[\tilde{\mathbf{X}}] = \mu, \mathbf{E}[\tilde{\mathbf{X}}\tilde{\mathbf{X}}^{\mathbf{T}}] = \mathbf{\Sigma} + \mu\mu^{\mathbf{T}}, \mathbf{P}(\tilde{\mathbf{X}} \geq \mathbf{0}) = \mathbf{1}\}.$$

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- **Optimize over a family of distributions with known moments. Use extremal distribution to predict choices!**
- **Compare Jensen:** $\inf_{\tilde{\mathbf{c}} \sim (\mu, \boldsymbol{\Sigma})^+} \mathbf{E}\left[Z(\tilde{\mathbf{c}})\right] \geq Z(\mu)$.

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Theorem (Natarajan-Teo-Zheng (2011))

$$\begin{array}{rl} Z_P = \max & \sum_{j=1}^{n} Y_{j,j} \\ s.t. & \mathbf{a_i^T X a_i} - \mathbf{2b_i a_i^T x} + \mathbf{b_i^2} = \mathbf{0},\ \forall \mathbf{i} = \mathbf{1}, \ldots, \mathbf{m} \\ & X_{j,j} = x_j,\ \forall j \in \mathcal{B} \\ & \begin{pmatrix} 1 & \mu^{\mathbf{T}} & \mathbf{x}^{\mathbf{T}} \\ \mu & \Sigma + \mu\mu^{\mathbf{T}} & Y^T \\ \mathbf{x} & Y & X \end{pmatrix} \succeq_{cp} 0, \quad \mathbf{x} \textit{ choice prob for worst ca} \end{array}$$

Define

$$A_1 = \begin{pmatrix} -I^+\hat{U} & I^+ & I^- & -I_{|\mathcal{G}|} & O_{|\mathcal{G}|^2} & 0 \\ \mathbf{1}_N^T U & \mathbf{0}_N^T & \mathbf{0}_N^T & \mathbf{0}_{|\mathcal{G}|}^T & \mathbf{0}_{|\mathcal{G}|}^T & 1 \\ J & O_{N^2} & O_{N^2} & O_{N\times|\mathcal{G}|} & O_{N\times|\mathcal{G}|} & 0 \\ O_{|\mathcal{G}|\times(N\times P)} & I^+ & I^- & I_{|\mathcal{G}|} & I_{|\mathcal{G}|} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} I_{N\times P} & O_{(N\times P)\times N} & O_{(N\times P)\times N} & O_{(N\times P)\times 2|\mathcal{G}|} & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} O_{(N\times P)^2} & \frac{U^T}{2} & O_{(N\times P)\times(N+2|\mathcal{G}|+1)} \\ \frac{U}{2} & O_{N^2} & O_{N\times(N+2|\mathcal{G}|+1)} \\ O_{(N+2|\mathcal{G}|+1)\times(N\times P)} & O_{(N+2|\mathcal{G}|+1)\times N} & O_{(N+2|\mathcal{G}|+1)^2} \end{pmatrix}$$

$$\boldsymbol{c} = \begin{pmatrix} \boldsymbol{e} \\ \mathbf{0}_N \\ \boldsymbol{v} \\ \mathbf{0}_{2|\mathcal{G}|} \\ 0 \end{pmatrix}$$

Consider following completely positive problem.

$$
\begin{array}{lllll}
f_2^{CP}(\boldsymbol{d}) = & \min & H \cdot X + \boldsymbol{c}^T \boldsymbol{p} & & \text{Dual variables} \\
& s.t. & A_1 \boldsymbol{p} = \begin{pmatrix} \mathbf{0}_{|\mathcal{G}|} \\ A \\ \mathbf{1}_N \\ B\mathbf{1}_{|\mathcal{G}|} \end{pmatrix} & & \boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}_1 \\ \pi_2 \\ \boldsymbol{\pi}_3 \\ \boldsymbol{\pi}_4 \end{pmatrix} \\
& & diag(A_1 X A_1^T) = \begin{pmatrix} \mathbf{0}_{|\mathcal{G}|} \\ A^2 \\ \mathbf{1}_N \\ B^2\mathbf{1}_{|\mathcal{G}|} \end{pmatrix} & & \boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{\phi}_1 \\ \phi_2 \\ \boldsymbol{\phi}_3 \\ \boldsymbol{\phi}_4 \end{pmatrix} \\
& & A_2 \boldsymbol{p} - diag(A_2 X A_2^T) = \mathbf{0}_{N \times P}, & & \boldsymbol{\kappa} \\
& & \begin{pmatrix} 1 & \boldsymbol{p}^T \\ \boldsymbol{p} & X \end{pmatrix} \succcurlyeq_{cp} 0 & & \rho
\end{array} \tag{4}
$$

The completely positive program problem (4) is equivalent to problem (3), i.e, $f_2(\boldsymbol{d}) = f_2^{CP}(\boldsymbol{d})$ for given $\boldsymbol{d}$.

Co-positive Cone Formulation

$$C(\boldsymbol{d}) = \begin{pmatrix} 0 & \frac{\boldsymbol{c}^T}{2} \\ \frac{\boldsymbol{c}}{2} & H \end{pmatrix}$$

$$M = \begin{pmatrix} \rho & \frac{\boldsymbol{\pi}^T A_1 + \boldsymbol{\kappa}^T A_2}{2} \\ \frac{A_1^T \boldsymbol{\pi} + A_2^T \boldsymbol{\kappa}}{2} & A_1^T \Lambda(\phi) A_1 - A_2^T \Lambda(\boldsymbol{\kappa}) A_2 \end{pmatrix}$$

The dual problem is

$$\begin{aligned} f_2^{CO}(\boldsymbol{d}) = \max \quad & \rho + A\pi_2 + \mathbf{1}_N^T \boldsymbol{\pi}_3 + B\mathbf{1}_{|\mathcal{G}|}^T \boldsymbol{\pi}_4 + A^2\phi_2 + \mathbf{1}_N^T \phi_+ B^2 \mathbf{1}_{|\mathcal{G}|}^T \phi_4 \\ \text{s. t.} \quad & C(\boldsymbol{d}) - M \succcurlyeq_{co} 0 \end{aligned}$$

Therefore, we have a first stage problem formulated as

$$\begin{aligned} f_1^{CO} = \quad \max \quad & \rho + A\pi_2 + \mathbf{1}_N^T \pi_3 + A^2\phi_2 + \mathbf{1}_N^T \phi_3 \\ \text{s. t.} \quad & \textstyle\sum_i d_i = D \\ & C(\boldsymbol{d}) - M \succcurlyeq_{co} 0 \\ & \boldsymbol{d} \geq \mathbf{0} \end{aligned} \qquad (5)$$

Dedicated Graph - No Redeployment

$N = 10, D = A = 10.$

Attacker's mixed strategies under dedicated structure is :

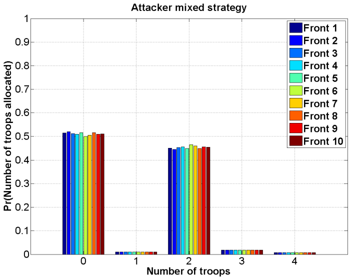


Figure: Attacker's mixed strategy under dedicated graph

This computation result recovers the close-form equilibrium we derived.

Numerical Experiments

2-Chain Structure (Long Cycle): Value of the game is 8.16. The number of troops allocated to each front by the defender are 1 in equilibrium in all cases; Attacker's mixed strategies under 2-chain structure is given in

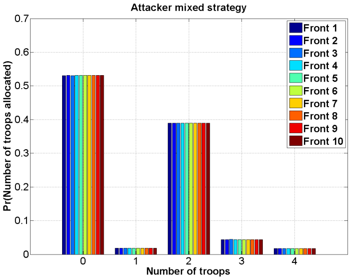
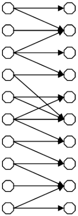


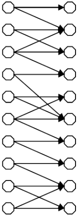
Figure: Attacker's mixed strategy under 2-chain graph

Assymetric Structure

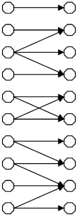
Which of the following is better for the defender?



: Asymmetric graph example 1



: Asymmetric graph example 2



: Asymmetric graph example 3

Figure: Asymmetric redeployment examples

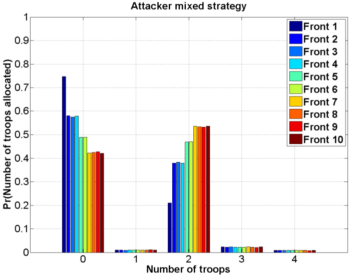
8.11 **8.18** **7.91**

For the third asymmetric graph, values of the game is 7.91. We obtained the number of troops allocated to each front by defender in Table 1.

Front	1	2	3	4	5	6	7	8	9	10
Troops	1.06	0.54	1.95	0.54	0.98	0.99	1.50	0.80	0.32	1.31

Table: Defender's strategy for asymmetric graph 3

Attacker's mixed strategies is given in Figure 8



Concluding Remarks

- This paper studies Blotto Game with sequential deployment and redeployment
- A modified CSF function has similar strategic implication as classical Blotto Game: The disadvantaged force picks battles to fight
- The value of flexibility (in redeployment)
- Copositive Conic Reformulation for the Defender's Solution
- Marginal Distribution of Attacker's Strategy.